3.8 Anti-Derivatives (Integration)

We have seen that if we have a function which gives us the position of an object, we can find the velocity by taking the derivative of the function. However, can we turn this around? Can we start with velocity and get an equation for the position? In this case, we want to find a function whose derivative is the equation for velocity, $v(t)$. If there is a function F such that $F'(x) = v(x)$, then we have a definition:

Definition 3.4. A function F is called an integral of f on an interval I if $F'(x) = f(x)$. Another term for an integral is an anti-derivative, which the text uses. Technically, the integral has a geometric definition which we cover in Chapter 4, but we will be able to use the terms interchangeably.

Let us examine $f(x) = x^4$. We want to find F such that $F'(x) = x^4$. We do know the power rule, and since g is a polynomial, this seems like a good place to start. We need a number to be killed off by the exponent and the exponent should be one bigger. A function that works is $F(x) = \frac{1}{5}x^5$, such that

$$\frac{d}{dx}\left[\frac{1}{5}x^5\right] = \frac{1}{5} \cdot 5x^{5-1} = x^4.$$

However, we also have $F(x) = \frac{1}{5}x^5 + 70$, and as such

$$\frac{d}{dx}\left[\frac{1}{5}x^5 + 70\right] = \frac{1}{5} \cdot 5x^{5-1} + 0 = x^4.$$

These are both integrals of f. In general, we could have any function $F(x) = \frac{1}{5}x^5 + c$, where c is any constant. This is a general integral of f. The question is, are there other general ones?

To answer this, think back to the Mean Value Theorem – a result from this is that is any two functions have the same derivative, then they must differ by a constant. Thus, if f and F are any two anti-derivatives of g, then

$$F'(x) = g(x) = f'(x),$$

and thus since they have the same derivative, F and f can differ only by a constant. Thus, there is only one general integral.

Theorem 3.7. *If F is an integral (anti-derivative) of f over an interval I, then the most general integral of f on I is*

$$F(x) + c,$$

where $F'(x) = f(x)$ and c is any arbitrary constant.

The only difference between different values of c is that the graph of F shifts up or down, but the general slope and shape of the graph is identical.

Example 3.29. Find the most general integral for each of the following functions:

$$f(x) = \sin(x).$$

If $F(x) = -\cos(x)$, then $F'(x) = -(-\sin(x)) = \sin(x)$. The most general integral function is $F(x) = -\cos(x) + C$.

$$h(x) = \frac{1}{x^2}.$$

If $F(x) = -x^{-1}$, then $F'(x) = -(-1)x^{-2} = \frac{1}{x^2}$. The most general integral function is

$$F(x) = \frac{-1}{x} + C$$

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3.8.1 Power Rule for Anti-Derivatives with Trig

If $f(x) = x^n$, then the general anti-derivative of f is

$$F(x) = \frac{x^{n+1}}{n+1} + C$$

Every single one of our differentiation formulas gives rise to an anti-differentiation formula, just when read backwards. Below we list some of the more common formulas for anti-derivatives, including a constant and a sum rule! (But no product or quotient...no no no):

Function	Antiderivative
$cf(x)$	$cF(x)$
$f(x) + g(x)$	$F(x) + G(x)$
$x^n, n \neq -1$	$\frac{1}{n+1}x^{n+1}$
$\cos(x)$	$\sin(x)$
$\sin(x)$	$-\cos(x)$
$\sec^2(x)$	$\tan(x)$
$\sec(x)\tan(x)$	$\sec(x)$
$\csc^2(x)$	$-\cot(x)$
$\csc(x)\cot(x)$	$-\csc(x)$

Example 3.30. Find a general function f such that

$$f'(x) = 3\cos(x) + \frac{3x^4 - \sqrt{x}}{x^2}.$$

We break the above function up so we do not have a complex fraction:

$$f'(x) = 3\cos(x) + \frac{3x^4}{x^2} - \frac{\sqrt{x}}{x^2} = 3\cos(x) + 3x^2 - x^{-3/2}.$$

Then, we can find an anti-derivative above by using reverse-derivative rules in the table above:

$$\begin{aligned} f(x) =& 3\sin(x) + 3 \cdot \frac{1}{2+1}x^{2+1} - \frac{1}{-3/2+1}x^{-3/2+1} + c \\ =& 3\sin(x) + x^3 + 2x^{-1/2} + c \end{aligned}$$

In Calculus, we have this situation an awful lot, where we want to find a function given some information about its derivative. The general solution usually involves some arbitrary constant, but we may be given some additional conditions to find the exact value of this constant.

Example 3.31. Find f if

$$f'(x) = 20\sin(x) + 5x^3,$$

and $f(0) = 3$.

The general antiderivative of

$$f'(x) = 20\sin(x) + 5x^3$$

is

$$f(x) = -20\cos(x) + \frac{5}{4}x^4 + C.$$

In order to find the exact value of c, we use the fact that $f(0) = 3$. We let $x = 0$ and solve for c:

$$\begin{aligned}
3 =& -20 \cdot \cos(0) + \frac{5}{4}(0)^4 + c \\
3 =& -20 + c \\
c =& 23
\end{aligned}$$

Example 3.32. Find f if

$$f''(x) = 5x^2 + 2x - 1,$$

$f'(0) = 2$ and $f(1) = 3$.

The general antiderivative of

$$f''(x) = 5x^2 + 2x - 1,$$

is

$$f'(x) = \frac{5}{3}x^3 + x^2 - x + C_1.$$

In order to find the exact value of C_1, we use the fact that $f'(0) = 2$. We let $x = 0$ and solve for C_1:

$$\begin{aligned}
2 =& \frac{5}{3}(0)^3 + 0^2 - 0 + C_1 \\
C_1 =& 2
\end{aligned}$$

Then, we have

$$f'(x) = \frac{5}{3}x^3 + x^2 - x + 2,$$

and we have general integral

$$f(x) = \frac{5}{12}x^4 + \frac{1}{3}x^3 - \frac{1}{2}x^2 + 2x + C_2.$$

In order to find the exact value of c_2, we use the fact that $f(1) = 3$. We let $x = 1$ and solve for C_2:

$$\begin{aligned}
3 =& \frac{5}{12}(1) + \frac{1}{3}(1) - \frac{1}{2}(1) + 2(1) + C_2 \\
3 =& \frac{5}{12} + \frac{4}{12} - \frac{6}{12} + \frac{24}{12} + C_2 \\
3 =& \frac{27}{12} + C_2 \\
C_2 =& \frac{9}{4} - 3 = \frac{-3}{4}
\end{aligned}$$

Then, we have

$$f(x) = \frac{5}{12}x^4 + \frac{1}{3}x^3 - \frac{1}{2}x^2 + 2x - \frac{3}{4}.$$

As we started this section, we talked about finding the position of an object if we start with the velocity. For that matter, we could start with the acceleration and go up from there.

Example 3.33. Suppose a velociraptor accelerates according to the function

$$a(t) = 12t - 8.$$

If the initial position is $s(0) = 10$ and the initial velocity if $v(0) = 1$, find the position function of the raptor $s(t)$.

The general antiderivative of

$$a(t) = 12t - 8$$

is

$$v(t) = \frac{12}{2}t^2 - 8t = 6t^2 - 8t + C_1.$$

In order to find the exact value of C_1, we use the fact that $v(0) = 1$. We let $t = 0$ and solve for C_1:

$$\begin{aligned} 1 =& 6(0)^2 - 8(0) + C_1 \\ C_1 =& 1 \end{aligned}$$

Then, we have

$$v(t) = 6t^2 - 8t + 1,$$

and we have general integral

$$s(t) = \frac{6}{3}t^3 - \frac{8}{2}t^2 + t + c_2 = 2t^3 - 4t^2 + t + c_2.$$

In order to find the exact value of c_2, we use the fact that $s(0) = 10$. We let $x = 0$ and solve for c_2, which gives

$$c_2 = 10.$$

Thus,

$$s(t) = 2t^3 - 4t^2 + t + 10.$$

Example 3.34. We throw a watermelon up in the air at an initial speed of 80ft/sec off the top of a 9 story building which is 160 feet off the ground. If we assume that the watermelon accelerates at -32 ft/sec^2, when does the watermelon reach its maximum height, and when does it hit the ground?

In order to find the maximum height, we need the velocity function $v(t)$ and to find when the ball hits the ground, we need the position function $s(t)$. We have

$$s''(t) = a(t) = -32.$$

Taking an antiderivative, we have

$$s'(t) = v(t) = -32t + C,$$

and we can find c by using the initial speed, $v(0) = 80$. Thus,

$$s'(t) = v(t) = -32t + 80.$$

Then, we can anti-differentiate again and get

$$s(t) = -16t^2 + 80t + C_1,$$

and to find C_1, we use the fact that $s(0) = 160$, so

$$s(t) = -16t^2 + 80t + 160.$$

Now, we can find the maximum height using the velocity function – the maximum height is reached when $v(t) = 0$:

$$\begin{aligned} v(t) &= 0 \\ -32t + 80 &= 0 \\ t &= \frac{80}{32} = \frac{5}{2} \end{aligned}$$

Thus, the maximum height is reached after $5/2$ seconds. In order to find when the ball hits the ground, we solve $s(t) = 0$:

$$\begin{aligned} s(t) &= 0 \\ -16t^2 + 80t + 160 &= 0 \\ t^2 - 5t - 10 &= 0 \\ t &= \frac{5 \pm \sqrt{25 - 4(1)(-10)}}{2(1)} = \frac{5 \pm \sqrt{65}}{2} \end{aligned}$$

Since $5-\sqrt{65} < 0$ and negative time makes no sense, we discard this solution. Thus, the watermelon hits the ground at

$$t = \frac{5+\sqrt{65}}{2} \approx 6.53 \text{ seconds.}$$

Example 3.35. Find f when $f''(x) = x - \sqrt{x}$

1. Rewrite f'' as $f''(x) = x - x^{1/2}$.

2. Find $f'(x)$

$$f'(x) = \frac{1}{2}x^2 - \frac{2}{3}x^{3/2} + C_1$$

3. Since we're not given any information about $f'(x)$, we move on to finding $f(x)$.

$$f(x) = \frac{1}{2} \cdot \frac{1}{3}x^3 - \frac{2}{3}x^{5/2} \cdot \frac{2}{5} + C_1x + C_2$$

$$f(x) = \frac{1}{6}x^3 - \frac{4}{15}x^{5/2} + C_1x + C_2$$