2.5 The Chain Rule

This is our last differentiation rule for this course. It's also one of the most used. The best way to memorize this (along with the other rules) is just by practicing until you can do it without thinking about it.

Ok, so what's the chain rule? It's the rule that allows us to differentiate a composition of functions. One way to think about a composition of functions to think of it having an 'outside' function and 'inside' function. For example,

1. $y = (3x^8 + \sqrt{x})^{25}$

   (a) The outside function: $f(x) = x^{25}$

   (b) The inside function: $g(x) = 3x^8 + \sqrt{x}$

   To check this, what is $f \circ g$?

$$\begin{aligned} f \circ g &= f\left(g(x)\right) \\ &= f\left(3x^8 + \sqrt{x}\right) \\ &= \left(3x^8 + \sqrt{x}\right)^{25} \end{aligned}$$

2. $y = \sqrt{x^3 + 1}$

   (a) The outside function: $f(x) = \sqrt{x}$

(b) The inside function: $g(x) = x^3 + 1$

(c) $f(g(x)) = \sqrt{x^3 + 1}$

3. $y = \cos^4(x)$

(a) Remember that $\cos^4(x)$ is a special notation. It represents $(\cos(x))^4$

(b) The outside function: $f(x) = x^4$

(c) The inside function: $g(x) = \cos(x)$

(d) $f(g(x)) = (\cos(x))^4$

4. $y = \sec^2(\sqrt{(}x))$

(a) A composition of functions doesn't have to be just a composition of two functions. In this example, it's a composition of three functions.

(b) Rewrite y as $y = (\sec(\sqrt{x}))^2$

(c) The outside function: $f(x) = x^2$

(d) 1st inside function: $g(x) = \sec(x)$

(e) 2nd inside function: $h(x) = \sqrt{x}$

(f) The composition is $y = f(g(h(x)))$

We went through all those examples because it's important you know how to identify the composition. What is the outside function? What is the inside function? Is there another inside function?

2.5.1 Chain Rule

If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $f(g(x))$ is differentiable and

$$\frac{d}{dx}\left[f\left(g(x)\right)\right] = f'\left(g(x)\right) \cdot g'(x)$$

Steps:

1. Differentiate the outside function f

2. Leave the inside function alone g

3. Multiply by the derivative of the inside function g'

Example 2.26. Find y' when $y = \left(x^3 + 5x\right)^5$

1. Identify the outside function f and the inside function g.

(a) Outside function: $f(x) = x^5$, with $f'(x) = 5x^4$

(b) Inside function: $g(x) = x^3 + 5x$, with $g'(x) = 3x^2 + 5$

2. Write out the chain rule formula.

$$y' = f'\left(g(x)\right) \cdot g'(x)$$

3. Fill in the parts (You won't be doing it like this once you get better at it)

$$\begin{aligned} y' &= f'\left(g(x)\right) \cdot g'(x) \\ &= 5\left(g(x)\right)^4 \cdot g'(x) \\ &= 5\left(x^3 + 5x\right)^4 \cdot \left(3x^2 + 5\right) \end{aligned}$$

Example 2.27. If $y = \cos^2(x)$, find y'.

1. Rewrite y as $y = (\cos(x))^2$

2. Identify the outside and inside functions.

(a) The outside function: $f(x) = x^2$, with $f'(x) = 2x$

(b) The inside function: $g(x) = \cos(x)$, with $g'(x) = -\sin(x)$

3. Use the chain rule

$$y' = 2\left(\cos(x)\right) \cdot - \sin(x)$$
$$y' = -2\cos(x)\sin(x)$$

Example 2.28. If $y = \sec(5x)$, find y'.

You may be surprised that many students get this wrong. It's because of the derivative of $\sec(x)$. Let's get started.

1. Identify the outside and inside functions

   (a) The outside function: $f(x) = \sec(x)$, with $f'(x) = \sec(x)\tan(x)$.

   (b) The inside function: $g(x) = 5x$, with $g'(x) = 5$.

2. Use the Chain Rule

$$\begin{aligned} y' &= f'\left(g(x)\right) \cdot g'(x) \\ &= \sec\left(g(x)\right)\tan\left(g(x)\right) \cdot g'(x) \\ &= \sec(5x)\tan(5x) \cdot 5 \\ &= 5\sec(5x)\tan(5x) \end{aligned}$$

Can you find y''? It's not that easy. Notice that it is now a product of functions. This means you'll have to do the product rule and the chain rule in the same problem.

Example 2.29. Find $\dfrac{d}{dx}\left[\cos(x^{-5}+\sin(x))\right]$

1. Identify the outside and inside functions

   (a) The outside function: $f(x) = \cos(x)$, with $f'(x) = -\sin(x)$

   (b) The inside function: $g(x) = x^{-5} + \sin(x)$, with $g'(x) = -5x^{-6} + \cos(x)$

2. When I do the chain rule, I say the following in the head,

   (a) Differentiate the outside function and leave the inside alone

   (b) Multiply by the derivative of the inside

3. Use the chain rule

$$y' \; = \; -\sin\left(x^{-5}+\sin(x)\right)\cdot\left(-5x^{-6}+\cos(x)\right)$$

So far we've differentiated a composition of two functions. Recall that a composition of functions can have any number of functions. Here's how the chain rule looks when you have a composition of three functions.

2.5.2 Chain Rule with a composition of three functions

$$\frac{d}{dx}\left[f\left(g\left(h(x)\right)\right)\right] = f'\left(g\left(h(x)\right)\right) \cdot g'(h(x)) \cdot h(x)$$

Example 2.30. Find y' when $y = \cos^5(\sqrt{x})$.

1. Anytime you're asked to differentiate a trig function with an exponent like this, rewrite it.

$$y = \left(\cos(\sqrt{x})\right)^5$$

2. Identify the inside and outside functions. I don't know if you can tell yet, but there's more than one inside function.

   (a) The outside function: $f(x) = x^5$, with $f'(x) = 5x^4$

   (b) The inside function: $g(x) = \cos(\sqrt{x})$. Note that $g'(x)$ requires the chain rule.

3. Use the chain rule

$$\begin{aligned}
y' &= 5(\cos(\sqrt{x})^4 \cdot \frac{d}{dx}\left[\cos(\sqrt{x})\right] \\
&= 5(\cos(\sqrt{x})^4 \cdot -\sin(\sqrt{x}) \cdot \frac{d}{dx}\left[\sqrt{x}\right] \\
&= 5(\cos(\sqrt{x})^4 \cdot -\sin(\sqrt{x}) \cdot -\frac{1}{2}x^{-1/2} \\
&= \frac{5}{2}x^{-1/2} \cdot (\cos(\sqrt{x}))^4 \cdot \sin(\sqrt{x})
\end{aligned}$$

Now it's time for some more complicated derivatives. These involve using the chain, product, and quotient rule at the same time.

Example 2.31. Find y' when $y' = \dfrac{(\cos(x)+x^3)^4}{\tan(x^2)}$.

1. Notice that when we differentiate the numerator $(\cos(x)+x^3)$ and the denominator $\tan(x^2)$, we will use the chain rule. But the overall function is a quotient. So which one goes first?

2. Since the main function is a quotient, we use the quotient rule.

3. Let's get started with the quotient rule.

$$y' = \frac{\tan(x^2)\cdot \frac{d}{dx}[(\cos(x)+x^3)^4] - (\cos(x)+x^3)^4 \cdot \frac{d}{dx}[\tan(x^2)]}{(\tan(x^2))^2}$$

4. I **STRONGLY** recommend showing all your work. Notice that all I have to do now is fill in those $\frac{d}{dx}$ with the correct derivatives. Breaking it up like this is very useful.

$$\begin{aligned}\frac{d}{dx}\left[(\cos(x)+x^3)^4\right] &= 4\left(\cos(x)+x^3\right)^3 \cdot \frac{d}{dx}\left[\cos(x)+x^3\right] \\ &= 4\left(\cos(x)+x^3\right)^3 \cdot \left(-\sin(x)+3x^2\right)\end{aligned}$$

$$\begin{aligned}\frac{d}{dx}\left[\tan(x^2)\right] &= \sec^2(x^2)\cdot \frac{d}{dx}\left[x^2\right] \\ &= \sec^2(x^2)\cdot 2x\end{aligned}$$

5. Now just replace the $\dfrac{d}{dx}$'s and we're done.

$$\begin{aligned} y' &= \frac{\tan(x^2)\cdot\dfrac{d}{dx}\left[(\cos(x)+x^3)^4\right]-(\cos(x)+x^3)^4\cdot\dfrac{d}{dx}\left[\tan(x^2)\right]}{(\tan(x^2))^2} \\ &= \frac{\tan(x^2)\cdot\left[4\left(\cos(x)+x^3\right)^3\cdot(-\sin(x)+3x^2)\right]-(\cos(x)+x^3)^4\cdot\left[\sec^2(x^2)\cdot 2x\right]}{(\tan(x^2))^2}\end{aligned}$$

Example 2.32. Find y' when $y=(x^3-2x^{11})^4\cdot(1-15x^2)^{100}$.

1. The overall function is a product of two functions $(x^3-2x^{11})^4$ and $(1-15x^2)^{100}$. We'll start with the product rule.

2. Product Rule

$$y' = (x^3-2x^{11})^4\cdot\frac{d}{dx}\left[(1-15x^2)^{100}\right]+(1-15x^2)^{100}\cdot\frac{d}{dx}\left[(x^3-2x^{11})^4\right]$$

3. We have two chain rules to work on now.

   (a) $\dfrac{d}{dx}\left[(1-15x^2)^{100}\right]$

$$\begin{aligned}\frac{d}{dx}\left[(1-15x^2)^{100}\right] &= 100(1-15x^2)^{99}\cdot\frac{d}{dx}\left[1-15x^2\right] \\ &= 100(1-15x^2)^{99}\cdot(-30x)\end{aligned}$$

(b) $\dfrac{d}{dx}\left[(x^3 - 2x^{11})^4\right]$

$$\begin{aligned} \frac{d}{dx}\left[(x^3 - 2x^{11})^4\right] &= 4(x^3 - 2x^{11})^3 \cdot \frac{d}{dx}\left[x^3 - 2x^{11}\right] \\ &= 4(x^3 - 2x^{11})^3 \cdot (3x^2 - 22x^{10}) \end{aligned}$$

4. Now put it all together

$$\begin{aligned} y' &= (x^3 - 2x^{11})^4 \cdot \frac{d}{dx}\left[(1 - 15x^2)^{100}\right] + (1 - 15x^2)^{100} \cdot \frac{d}{dx}\left[(x^3 - 2x^{11})^4\right] \\ &= (x^3 - 2x^{11})^4 \cdot \left[100(1 - 15x^2)^{99} \cdot (-30x)\right] + (1 - 15x^2)^{100} \cdot \left[4(x^3 - 2x^{11})^3 \cdot (3x^2 - 22x^{10})\right] \end{aligned}$$

You have some common factors you can pull out. I'll leave that to you as an exercise. At some point in the near future, you will have to know how to factor this.

Example 2.33. Find y' when $y = \sqrt{\dfrac{x^2 + 1}{x^3 - 2\cos(x)}}$

1. Make sure everything is written in the correct form.

$$y = \left(\frac{x^2 + 1}{x^3 - 2\cos(x)}\right)^{1/2}$$

2. This is a chain rule first, then a quotient rule.

3. Start the chain rule

$$y' = \frac{1}{2}\left(\frac{x^2+1}{x^3-2\cos(x)}\right)^{-1/2} \cdot \frac{d}{dx}\left[\frac{x^2+1}{x^3-2\cos(x)}\right]$$

4. So now we just have to find $\dfrac{d}{dx}\left[\dfrac{x^2+1}{x^3-2\cos(x)}\right]$

$$\frac{d}{dx}\left[\frac{x^2+1}{x^3-2\cos(x)}\right] = \frac{(x^3-2\cos(x))\cdot(2x)-(x^2+1)\cdot(3x^2+2\sin(x))}{(x^3-2\cos(x))^2}$$

5. Put it all together

$$\begin{aligned} y' &= \frac{1}{2}\left(\frac{x^2+1}{x^3-2\cos(x)}\right)^{-1/2} \cdot \frac{d}{dx}\left[\frac{x^2+1}{x^3-2\cos(x)}\right] \\ &= \frac{1}{2}\left(\frac{x^2+1}{x^3-2\cos(x)}\right)^{-1/2} \cdot \frac{d}{dx}\left[\frac{(x^3-2\cos(x))\cdot(2x)-(x^2+1)\cdot(3x^2+2\sin(x))}{(x^3-2\cos(x))^2}\right] \end{aligned}$$

Again, I'm not focused with simplifying at the moment. You will have the rest of the semester simplifying these things. Plus, I'm also giving examples that are nasty to simplify.

Example 2.34. If $h(x) = g(f(x))$, $m(x) = g(x^3)$, $f(2) = 8$, $g'(8) = -2$, and $f'(2) = 3$, find $h'(2)$.

1. Find $h'(x)$ using the chain rule

$$h'(x) = g'(f(x)) \cdot f'(x)$$

2. Plug in $x = 2$

$$\begin{aligned} h'(2) &= g'(f(2)) \cdot f'(2) \\ &= g'(8) \cdot 3 \\ &= -2 \cdot 3 \\ &= -6 \end{aligned}$$

Example 2.35. Using the previous example's information, find $m'(2)$.

1. Find $m'(x)$

$$m'(x) = g'(x^3) \cdot \frac{d}{dx}\left[x^3\right] = g'(x^3) \cdot 3x^2$$

2. Plug in $x = 2$

$$\begin{aligned} m'(2) &= g'(2^3) \cdot 3(2)^2 \\ &= g'(8) \cdot 12 \\ &= -2 \cdot 12 \\ &= -24 \end{aligned}$$

As I stated before, practice these rules. In your textbook there are plenty of problems to practice.