1.10 Continuity

Definition 1.5. A function is continuous at $x = a$ if

1. $f(a)$ exists

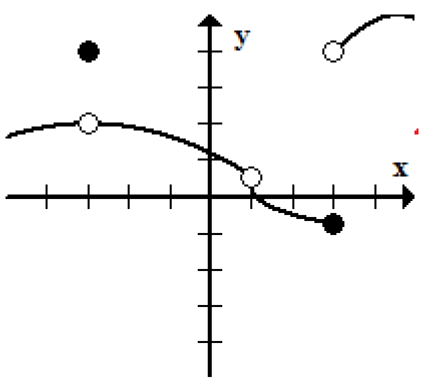
2. $\lim_{x \to a} f(x)$ exists

3. $\lim_{x \to a} f(x) = f(a)$

If any of these conditions fail, f is discontinuous.

Note: From algebra you probably said a function is not continuous if when tracing the graph with your pencil, you have to lift your pencil off the paper. Even though this isn't a formal way of defining continuity, it's a good way of looking at it.

Example 1.31. Where is f discontinuous? Continuous?



Let's start with finding all the places f is discontinuous. If f is discontinuous, it must fail at least one condition.

1. Discontinuous at $x = -2$.

   (a) $f(-2)$ exists? Yes, and $f(-2) = 4$. Check!

   (b) Does $\lim_{x \to -2} f(x)$ exist? Yes, and $\lim_{x \to -2} f(x) = 2$. Check!

   (c) Does $\lim_{x \to -2} f(x) = f(-2)$? No!

   It fails condition (3). Therefore, f is not continuous at $x = -2$.

2. Discontinuous at $x = 1$.

   (a) $f(1)$ exists? No. There is no y-value for $x = 1$.

   It fails condition (1). Therefore, f is not continuous at $x = 1$.

3. Discontinuous at $x = 3$.

   (a) $f(3)$ exists? Yes, and $f(3) = -1$. Check!

   (b) Does $\lim_{x \to 1} f(x)$ exist? No. Let's look at the left and right hand limits.

      i. $\lim_{x \to 3^-} f(x) = -1$.

      ii. $\lim_{x \to 3^+} f(x) = 4$.

Since $\lim_{x\to 3^-} f(x) \neq \lim_{x\to 3^+} f(x)$, the general limit $\lim_{x\to 3} f(x)$ does not exist.

It fails condition (2). Therefore, f is not continuous at $x = 3$.

There are no other problem points on the graph. Use your algebra definition of continuity and trace the graph. You only have to lift your pencil off the paper at $x = -2$, $x = 1$, and $x = 3$.

This means f in continuous everywhere except $x = -2$, $x = 1$, and $x = 3$. We can write it in interval notation: $(-\infty, -2) \cup (-2, 1) \cup (1, 3) \cup (3, \infty)$.

So what type of functions do we typically have continuity issues?

1. Piece-wise functions

2. Rational functions

Example 1.32. Let $f(x) = \begin{cases} 5 - 4x, & x \leq 1 \\ \sqrt{x} + 1, & x > 1 \end{cases}$

I hope at this point we know that polynomials, radicals, and rational functions are continuous everywhere on their domain.

1. $5 - 4x$ is continuous everywhere, especially when $x < 1$
2. $\sqrt{x}+1$ is continuous everywhere on its domain $(x \geq 0)$. So it's continuous when $x > 1$.

The only problem we may have is when we jump from $5-4x$ to $\sqrt{x}+1$ at $x=1$. Let's go through the conditions.

1. Does $f(1)$ exist? Yes and $f(1) = 5-4(1) = 1$.

2. Does $\lim\limits_{x\to 1} f(x)$ exist?

   Here we have to be careful. A limit exists when its left and right hand limits exist and equal each other. We use a different function for the left hand limit than the right hand limit.

   (a) Left Hand Limit:

   $$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 5-4x = 5-4(1) = 1$$

   (b) Right Hand Limit:

   $$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \sqrt{x}+1 = \sqrt{1}+1 = 2$$

Since $\lim\limits_{x\to 1} f(x)$ does not exist, $f(x)$ is not continuous at $x=1$.

Example 1.33. Let $f(x) = \dfrac{x^2-x-2}{x-2}$. Where is f continuous?

It's easier to look for when f is discontinuous. You should be able to see right away that $x \neq 2$. Also, note that $f(x)$ is a rational function. Rational functions are continuous everywhere on their domain. Since the only x-value we aren't allowed to plug in is $x = 2$, we conclude $f(x)$ is continuous everywhere except $x = 2$.

$$(-\infty, 2) \cup (2, \infty)$$

Example 1.34. Consider the piece-wise function $f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2}, & x \neq 2 \\ 3, & x = 2 \end{cases}$

Where is f continuous?

Note that $f(x)$ is the rational function $\dfrac{x^2 - x - 2}{x - 2}$ everywhere except $x = 2$. So just from that, we know $f(x)$ is continuous everywhere except possibly at $x = 2$. Let's go through the continuity conditions.

1. Does $f(2)$ exist? Yes and $f(2) = 3$.

2. Does $\lim_{x \to 2} f(x)$ exist?

   This is a problem from a couple sections ago. When trying to evaluate a general limit, just plug in $x = 2$ and see what happens.

$$\frac{2^2 - 2 - 2}{2 - 2} = \frac{0}{0}$$

   Ok, so we get $\frac{0}{0}$. That means we should try one of our limit techniques. In this case it's **limits with cancellation (factoring)**

$$\begin{aligned}\lim_{x\to 2}\frac{x^2-x-2}{x-2} &= \lim_{x\to 2}\frac{(x-2)(x+1)}{x-2}\\ &= \lim_{x\to 2} x+1\\ &= 3\end{aligned}$$

So $\lim_{x\to 2} f(x)$ exists.

3. Does $\lim_{x\to 2} f(x) = f(2)$?

   Yes. Since $f(x)$ satisfies all three conditions at $x = 2$, we conclude $f(x)$ is continuous at $x = 2$.

Therefore, $f(x) = \begin{cases} \dfrac{x^2-x-2}{x-2}, & x \neq 2 \\ 3, & x = 2 \end{cases}$ is continuous everywhere.

Example 1.35. Is $f(x) = \begin{cases} \dfrac{\sqrt{4+x}-2}{x}, & x \neq 0 \\ 4, & x = 0 \end{cases}$ continuous at $x = 0$?

1. Does $f(0)$ exist? Yes, $f(0) = 4$.

2. Does $\lim_{x\to 0} f(x)$ exist?

$$\begin{aligned}
\lim_{x\to 0}\frac{\sqrt{4+x}-2}{x} &= \lim_{x\to 0}\frac{\sqrt{4+x}-2}{x}\cdot\frac{\sqrt{4+x}+2}{\sqrt{4+x}+2}\\
&= \lim_{x\to 0}\frac{(4+x)-4}{x(\sqrt{4+x}+2)}\\
&= \lim_{x\to 0}\frac{x}{x(\sqrt{4+x}+2)}\\
&= \lim_{x\to 0}\frac{1}{\sqrt{4+x}+2}\\
&= \frac{1}{\sqrt{4+0}+2}\\
&= \frac{1}{4}
\end{aligned}$$

So $\lim_{x\to 0} f(x)$ exists.

3. Does $\lim_{x\to 0} f(x) = f(0)$? No.

$f(x)$ violates condition (3), so $f(x)$ is not continuous at $x=0$.

Example 1.36. Let $f(x) = \dfrac{\sin(x)}{2\cos(x)-1}$. Where is $f(x)$ continuous?

You should already know that $\sin(x)$ and $\cos(x)$ are continuous everywhere. The only thing we're worried about is when the denominator is zero. So let's solve the following equation:

$$2\cos(x)-1=0$$

Let's get started.

$$\begin{aligned}
2\cos(x)-1 &= 0\\
2\cos(x) &= 1\\
\cos(x) &= 1/2
\end{aligned}$$

So when does $\cos(x) = 1/2$?

$$x = \frac{\pi}{3}$$

and

$$x = \frac{5\pi}{3}$$

But this is only between $[0, 2\pi]$. Every full revolution around the unit circle starting with $x = \frac{\pi}{3}$ or $x = \frac{5\pi}{3}$ is also a solution to the equation.

Therefore, $f(x) = \dfrac{\sin(x)}{2\cos(x) - 1}$ is continuous everywhere except when

$$x = \frac{\pi}{3} \pm 2n\pi, n = 0, 1, 2, ...$$
$$x = \frac{5\pi}{3} \pm 2n\pi, n = 0, 1, 2, ...$$

Example 1.37. Let $f(x) = \begin{cases} (x-a)^2, & x < 0 \\ \cos(x), & 0 \leq x < \pi/2 \end{cases}$

What value of a would make $f(x)$ continuous at $x = 0$?

In order to be continuous at $x = 0$, $f(x)$ needs to satisfy the three conditions of continuity.

1. Does $f(0)$ exist?

   Yes, $f(0) = \cos(0) = 1$

2. Does $\lim_{x\to 0} f(x)$ exist?

   (a) Left Hand Limit

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (x-a)^2 = (0-a)^2 = a^2$$

   (b) Right Hand Limit

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \cos(x) = \cos(0) = 1$$

So if $f(x)$ is continuous, the left and right hand limits must equal. This means $a^2 = 1$

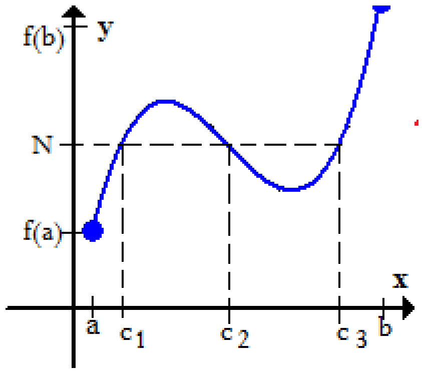
Solving this, we get $a = -1$ or $a = 1$.

1.10.1 Intermediate Value Theorem

Theorem 1.3 (Intermediate Value Theorem)**.** *Suppose f is continuous on $[a, b]$ and let N (a y-value) be any real number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$, there exists c in (a, b) such that*

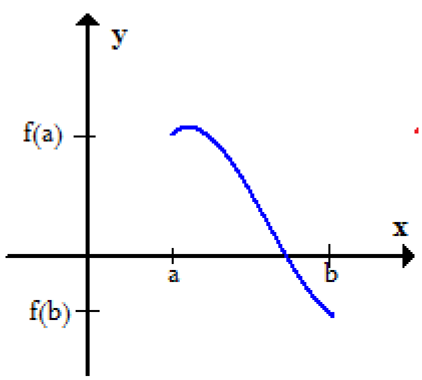
$$f(c) = N$$

Let's take a look at a graph.



In this example, there are actually three c values that satisfy $f(c_i) = N$. The point is if you have a continuous function then you must hit every y-value between $f(a)$ and $f(b)$. This means for every y-value between $f(a)$ and $f(b)$ there must be at least one x-value between a and b that gets you the y-value.

So why do we care about this theorem? It's extremely useful when you're trying to find roots of a function. Just let $N = 0$. Take a look at the following graph.



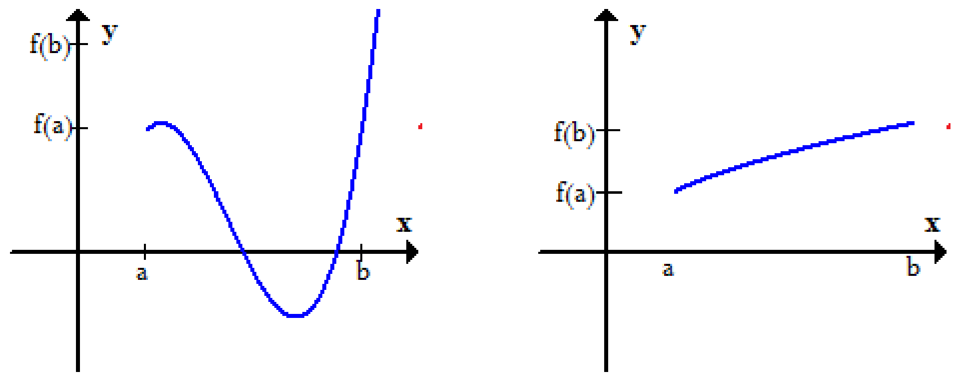
Since f is continuous, $f(a) > 0$ and $f(b) < 0$, there must be some c, (x-value) in (a, b) such that $f(c) = 0$.

Another way of wording it is,

If f is continuous and $f(a)$ and $f(b)$ have opposite signs, then a root must exist in the interval (a, b).

Concerns about trying to use IVT

What happens if $f(a)$ and $f(b)$ have the same sign, does that mean there won't be a root in the interval (a, b). To answer this, look at these two graphs.



In both of these examples, $f(a)$ and $f(b)$ are positive. In the first graph, however, we do see roots. In the second graph, we don't have any roots. So what can I take away from this?

If you don't satisfy the conditions to a theorem (in this case $f(a)$ and $f(b)$ have opposite signs), you cannot say anything about the conclusion to the theorem. You cannot conclude there are no roots or that there are roots. Theorems work only when you satisfy their conditions. Don't satisfy the conditions, you know absolutely knowing about the conclusion.

Example 1.38. Show there is a solution to $x = \sin(x)$.

1. If we want to use the Intermediate Value Theorem, we need a function.

$$\text{Let } f(x) = x - \sin(x)$$

2. Choose an interval. Sometimes the interval is given. If it's not given, try something simple like $(0, 1)$ or $(-2, 2)$.

3. I'll try the interval $(-1, 1)$.

   - $f(-1) = -1 - \sin(-1) = -0.15853$
   - $f(1) = 1 - \sin(1) = 0.15853$

4. We should also note that $f(x) = x - \sin(x)$ is continuous on the interval $(-1, 1)$.

5. So have we satisfied the requirements we need?

   (a) $f(x)$ is continuous on the interval? Check!

   (b) $f(a)$ and $f(b)$ have opposite signs? Check!

   Therefore, somewhere in the interval $(-1, 1)$ the function $f(x)$ must cross the x-axis. Conclusion: A root must exist in the interval $(-1, 1)$.