4.2 Definite Integral

When we compute the area under a curve, we obtained a limit of the form

$$\lim_{n\to\infty} \sum_{x=1}^{n} f(x_i^*)\Delta x.$$

This same limit shows up when we consider finding the distance given the velocity, and it turns our that this limit shows up in plenty of other situations. As it will appear for the next, ohhh, forever, we give it a special definition:

Definition 4.2. If f is a function defined over the interval $[a, b]$, we divide $[a, b]$ into n subintervals of equal width $\Delta x = \frac{b-a}{n}$. We let $x_0 = a, x_a, x_2, ..., x_{n-1}, x_n = b$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, ..., x_n^*$ be any sample points in the subintervals, so $x_i^* \in [x_{i-1}, x_i]$.

Then, the **definite integral** of f from a to b is

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x,$$

$$\Delta x = \frac{b-a}{n}$$

$$x_i = a + i\Delta x \text{ for right-hand endpoints}$$

provided that this limit exists and gives the same value for all possible choices of sample points. If this limit does exist, we say that f is **integrable** over $[a, b]$.

For notation,

$$\int_a^b f(x)dx,$$

we say that $\int$ is the integral sign, a is the lower limit and b is the upper limit of integration. The dx indicates that the independent variable is x – all other variables may be treated as constants. Also, the integral $\int_a^b f(x)dx$ is a number – it doesn not depend on x, and there is nothing special about x.

For the function f, we have

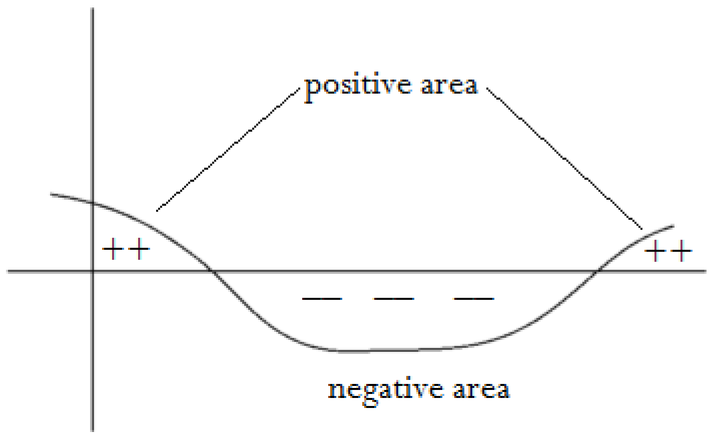
$$\int_a^b f(x)dx = \int_a^b f(t)dt = \int_a^b f(y)dy,$$

for any variable as a placeholder. Lastly, we say that the sum

$$\sum_{i=1}^{n} f(x_i^*)\Delta x$$

is called a Riemann sum. If our function of interest, $f(x)$, is always non-negative, then we can treat the Riemann sum as a sum of areas of rectangles. Since the definite integral is the limit of a Riemann sum, a definite integral is the limit of the sum of area of rectangles, and thus the area under the curve $f(x)$ from $x = a$ to $x = b$.

If we consider a function f that does take on negative values, such as a function like the one below,



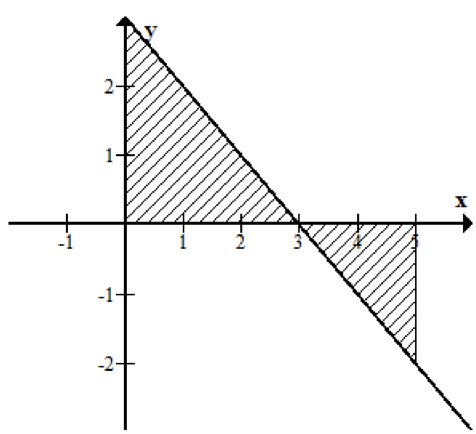
we cannot just take the heights of all the approximating rectangles – some of these heights would be negative, and that wouldn't make a whole lot of sense to have negative area. The way we can get around this is to consider the net area – we find the area of the curve above the x-axis. Then, separately, find the area of the curve under the x-axis. We then add these two areas together. So if A_1 is the area above the x-axis but under $f(x)$ computed by base times height, $\Delta x \cdot f(x_i^*)$ and A_2 is the area under the x-axis but above $f(x)$, also computed by base times height, $\Delta x \cdot f(x_i^*)$, we have

$$\text{Area of shaded region } = A_1 - A_2,$$

since A_2 is negative.

Example 4.6. Evaluate $\displaystyle\int_0^5 (3-x)\,dx$ and the area of the shaded region.

1. Let's take a look at the graph



Do you see how some of the shaded region is above the x-axis and some of it is below. When we evaluate $\displaystyle\int_0^5 (3-x)\,dx$, it would consider the area below negative. We can do it two different ways at this point. Let's do it without calculus.

2. Do you see how the two shaded regions are triangles? Guess what? We know the formula for the area of a triangle, $A = \frac{1}{2}bh$.

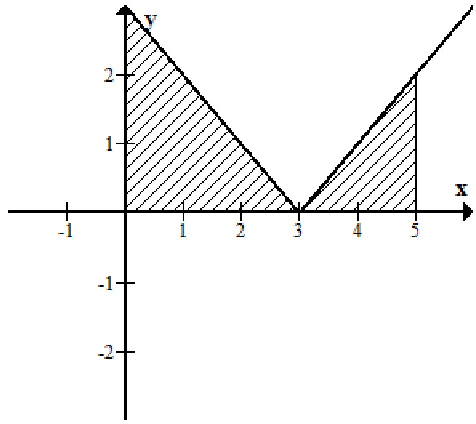
$$\text{Area of top triangle: } A = \frac{1}{2}(3)(3) = \frac{9}{2}$$
$$\text{Area of bottom triangle: } A = \frac{1}{2}(2)(2) = 2$$

So our integral is

$$\int_0^5 (3-x)\,dx = \frac{9}{2} - 2 = \frac{5}{2} = 2.5$$

3. Now if you want area of the shaded region (pretending like the area below the x-axis is also positive), then what we really want to evaluate is

$$\int_0^5 |3-x|\,dx$$



So the area of the shaded region is

$$\int_0^5 |3-x|\,dx = \frac{9}{2} + 2 = \frac{13}{2} = 6.5$$

Theorem 4.1. *If f is continuous on $[a,b]$, or if f has only a finite number of jump discontinuities, then f is integrable on $[a,b]$ – meaning that the definite integral $\int_a^b f(x)dx$ exists.*

This theorem is NOT easy to prove, by any stretch of the word easy. We do not do it here, but it does tell us something. It says that there are functions which are not integrable.

In order to simplify the calculations, we can choose specific sample points, and the right most endpoints are as good as any other. So, we have a half definition, half theorem:

Defi-thereom: If f is integrable on $[a,b]$, then

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

where

$$\Delta x = \frac{b-a}{n}$$

and $x_i = a + i\Delta x$.

In order to evaluate these integrals, we have to be able to work with some very commonly seen sums – the following three equations will be insanely valuable in doing this:

$$\begin{aligned}\sum_{i=1}^{n} 1 &= n \\ \sum_{i=1}^{n} i &= \frac{n(n=1)}{2} \\ \sum_{i=1}^{n} i^2 &= \frac{n(n+1)(2n+1)}{6} \\ \sum_{i=1}^{n} i^3 &= \left(\frac{n(n+1)}{2}\right)^2\end{aligned}$$

The remaining rules that will help evaluate sums are very similar to rules that let us evaluate limits:

$$\begin{aligned}\sum_{i=1}^{n} c &= nc \\ \sum_{i=1}^{n} ca_i &= c\sum_{i=1}^{n} a_i \\ \sum_{i=1}^{n}(a_i + b_i) &= \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i \\ \sum_{i=1}^{n}(a_i - b_i) &= \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i\end{aligned}$$

Example 4.7. Let's find the Riemann sum for $f(x) = 3 - x$ by taking right-hand endpoints over the interval [0,5]. By the way, we know the answer should be $\frac{5}{2}$. I'll take you through the procedure. Do this whenever you're asked to find evaluate the Riemann Sum.

1. Find Δx:

$$\Delta x = \frac{b-a}{n} = \frac{5-0}{n} = \frac{5}{n}$$

2. Find x_i:

$$x_i = a + i\Delta x = 0 + i \cdot \frac{5}{n} = \frac{5i}{n}$$

3. Find $f(x_i)$:

$$\begin{aligned} f(x_i) &= 3 - x_i \\ &= 3 - \left(\frac{5i}{n}\right) \end{aligned}$$

4. Find A_i:

$$\begin{aligned} A_i &= f(x_i) \cdot \Delta x \\ &= \left(3 - \frac{5i}{n}\right) \cdot \frac{5}{n} \\ &= \frac{15}{n} - \frac{25i}{n^2} \end{aligned}$$

5. Find $\sum_{i=1}^{n} A_i$:

$$\begin{aligned} \sum_{i=1}^{n} A_i &= \sum_{i=1}^{n} \frac{15}{n} - \frac{25i}{n^2} \\ &= \frac{15}{n}\sum_{i=1}^{n} 1 - \frac{25}{n^2}\sum_{i=1}^{n} i \\ &= \frac{15}{n} \cdot n - \frac{25}{n^2} \cdot \left(\frac{n(n+1)}{2}\right) \\ &= 15 - \frac{25n(n+1)}{2n^2} \end{aligned}$$

6. Our final step is take the limit as $n \to \infty$.

$$\text{True Area } = \lim_{n\to\infty} 15 - \frac{25n(n+1)}{2n^2} = 15 - \frac{25}{2} = 2.5$$

which is exactly what we got from the before.

Example 4.8. Find the Riemann sum for $f(x) = 2x^3 - 4x$ by taking sample points to be right endpoints, where $a = 0$, $b = 2$ and $n = 4$.

With $n = 4$, the interval width is

$$\frac{2-0}{4} = \frac{1}{2},$$

and the right endpoints are $x_1 = .5$, $x_2 = 1$, $x_3 = 1.5$ and $x_4 = 2$. The Riemann sum is

$$\begin{aligned} R_4 =& \sum_{i=1}^{4} f(x_i)\Delta x \\ =& \Delta x \left(f(0.5) + f(1) + f(1.5) + f(2) \right) \\ =& \frac{1}{2}\left(\left(\frac{1}{4} - 2\right) + (2-4) + \left(\frac{27}{4} - 6\right) + (16-8) \right) \\ =& \frac{5}{2} \end{aligned}$$

Note that since this function does dip negative, this value is NOT the approximation for the area under the curve. However, it does represent the difference in positive and negative areas of the approximating rectangles of the curve. But, this is just an approximation. Now, let's evaluate

$$\int_0^2 2x^3 - 4x dx.$$

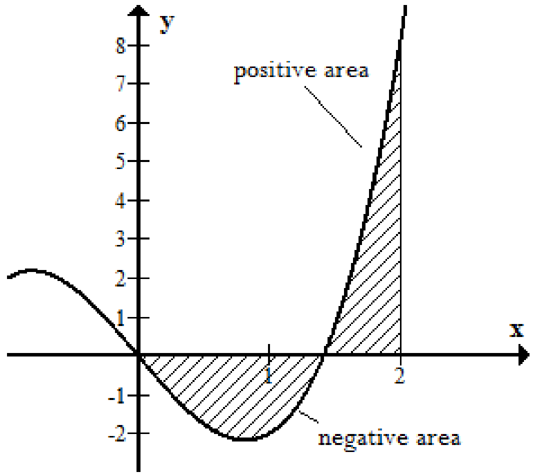
With n subintervals, we have

$$\Delta x = \frac{b-a}{n} = \frac{2}{n}.$$

and we have $x_0 = 0$, $x_1 = 2/n$, $x_2 = 4/n$, and in general, $x_i = 2i/n$. We use right endpoints and our theorem to let us evaluate

$$
\begin{aligned}
\int_0^2 2x^3 - 4x dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{2i}{n}\right) \cdot \frac{2}{n} \\
&= \lim_{n\to\infty} \frac{2}{n} \cdot \sum_{i=1}^{n} \left(2\left(\frac{2i}{n}\right)^3 - 4\left(\frac{2i}{n}\right)\right) \\
&= \lim_{n\to\infty} \frac{2}{n} \cdot \sum_{i=1}^{n} \left(\frac{16i^3}{n^3} - \frac{8i}{n}\right) \\
&= \lim_{n\to\infty} \frac{2}{n} \cdot \left(\frac{16}{n^3}\sum_{i=1}^{n} i^3 - \frac{8}{n}\sum_{i=1}^{n} i\right) \\
&= \lim_{n\to\infty} \frac{2}{n} \cdot \left(\frac{16}{n^3} \cdot \left(\frac{n(n+1)}{2}\right)^2 - \frac{8}{n} \cdot \left(\frac{n(n+1)}{2}\right)\right) \\
&= \lim_{n\to\infty} \frac{2}{n} \cdot \left(\frac{4(n+1)^2}{n} - 4(n+1)\right) \\
&=8 \lim_{n\to\infty} \frac{(n+1)^2}{n^2} - \frac{n+1}{n} \\
&=8 \cdot \lim_{n\to\infty} \frac{1 - 2/n - 1/n^2}{1} - \frac{1+1/n}{1} \\
&=8 \cdot (1-1) \\
&=0
\end{aligned}
$$

Clearly, this integral cannot be interpreted as the area under a curve, since this curve clearly does not have 0 area. However, it can be interpreted as a difference between the positive and negative area. If we graph the function, we should see that the positive and negative areas are identical:



This is NOT how we will be computing these soon. But just like as with derivatives, we have to go through it the long way first before we get to the shortcuts.

Example 4.9. Set up an expression for $\int_3^7 x^7\,dx$ as a limit of sums. Do not evaluate.

We let $f(x) = x^7$, $a = 3$ and $b = 7$. Thus,

$$\Delta x = \frac{7-3}{n} = \frac{4}{n}.$$

We have $x_1 = 3 + 1 \cdot \frac{4}{n}$, $x_2 = 3 + 2 \cdot \frac{4}{n}$, ..., and we have a generic term

$$x_i = 3 + \frac{4i}{n}.$$

By our theorem, we have

$$\begin{aligned}
\int_3^7 x^7 dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(3 + \frac{4i}{n}\right) \cdot \frac{4}{n} \\
&= \lim_{n\to\infty} \frac{4}{n} \sum_{i=1}^{n} \left(3 + \frac{4i}{n}\right)^7
\end{aligned}$$

This is not very easy. Soon enough we'll get to the shortcuts that allow us to evaluate this quickly. By the way,

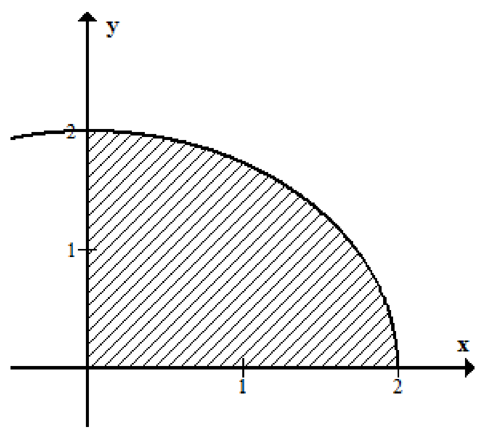
$$\int_3^7 x^7\,dx = 719780$$

Example 4.10. Evaluate

$$\int_0^2 \sqrt{4-x^2}dx$$

by interpreting the integral as an area.

Since $f(x) = \sqrt{4-x^2} \geq 0$ for x in $[0,2]$, we can interpret the integral as the area under $y = \sqrt{4-x^2}$ from 0 to 2. We can rewrite this as $y^2 = 4 - x^2$, which gives $x^2 + y^2 = 4$, which shows that the graph of f is part of a circle with radius $\sqrt{4} = 2$. Since our part of the circle is stuck in Quadrant I, we have a quarter circle. Thus, we take a quarter of the area:

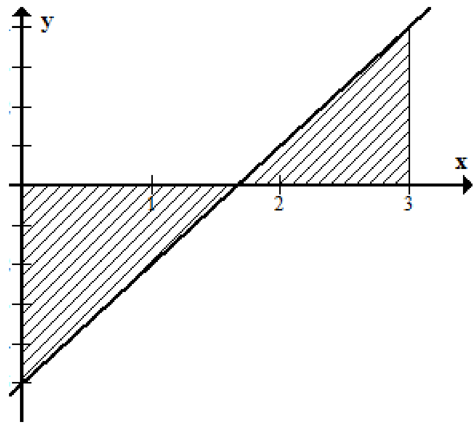


$$A = \frac{1}{4}\left(\pi \cdot r^2\right) = \frac{1}{4}\left(\pi \cdot 2^2\right) = \pi.$$

Example 4.11. Evaluate

$$\int_0^3 3x - 5\ dx.$$

We are benefited by graphing this first:



In order to find the integral, note that we get positive values for the triangle above the x-axis and negative values from the triangle below the x-axis. We find the areas of these two triangles, and subtract. First, find the x-intercept of $3x - 5 = 0$, which gives $x = \frac{5}{3}$. The upper triangle then has base equal to $3 - \frac{5}{3} = \frac{4}{3}$, with a height of $f(3) = 3(3) - 5 = 4$. Thus, the upper triangle has area

$$A_T = \frac{1}{2} \cdot \frac{4}{3} \cdot 4 = \frac{8}{3}.$$

Similarly, the lower triangle, which has base $\frac{5}{3}$ and height $f(0) = -5$ has area

$$A_B = \frac{1}{2} \cdot \frac{5}{3} \cdot |-5| = \frac{25}{6}.$$

Thus, the integral is

$$\int_0^3 3x - 5dx = A_T - A_B = \frac{8}{3} - \frac{25}{6} = \frac{-9}{6} = \frac{-3}{2}.$$

4.2.1 Midpoint Rule

These are all good integrals, but for some we still need to approximate, anything with a curved side, as a matter of fact, which is not part of a circle. We so far have chosen just left and right endpoints, and we saw in Section 4.1 how those gave us rather bad approximations. Better, perhaps, to use the midpoint of the intervals!

Definition 4.3 (The Midpoint Rule:)**.**

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(\bar{x}_i)\Delta x = \Delta x\left(f(\bar{x}_1) + f(\bar{x}_2) + \ldots + f(\bar{x}_n)\right),$$

where

$$\Delta x = \frac{b-a}{n}$$

and

$$\bar{x_i} = \frac{1}{2}(x_{i-1} + x_i),$$

which is the midpoint of the interval $[x_{i-1}, x_i]$.

Example 4.12. Use the midpoint rule with $n = 4$ to approximate

$$\int_0^1 \frac{1}{2x}\, dx.$$

We start with $\Delta x = \frac{1-0}{4} = \frac{1}{4}$. The righthand endpoints of the 4 intervals are $x_1 = 1/4, x_2 = 1/2, x_3 = 3/4, x_4 = 1$. To find the midpoints, add consecutive pairs and divide by 2:

$$\begin{aligned}
\bar{x_1} =& \frac{0+1/4}{2} = \frac{1}{8} \\
\bar{x_2} =& \frac{1/4+1/2}{2} = \frac{3}{8} \\
\bar{x_3} =& \frac{1/2+3/4}{2} = \frac{5}{8} \\
\bar{x_4} =& \frac{3/4+1}{2} = \frac{7}{8}
\end{aligned}$$

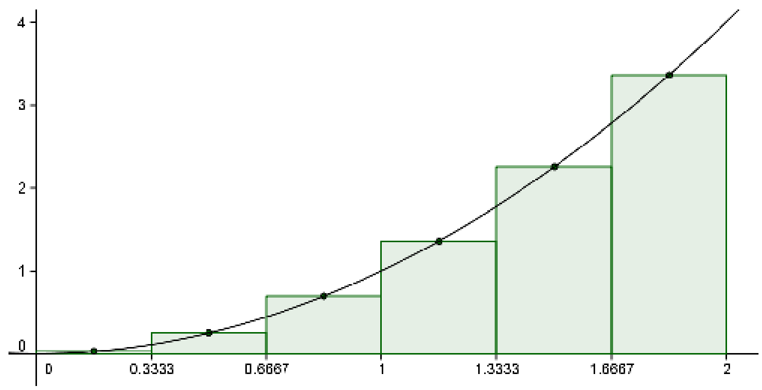
Thus, the midpoint rule gives

$$\begin{aligned}
\int_0^1 \frac{1}{2x} dx =& \Delta x \left(f(1/8) + f(3/8) + f(5/8) + f(7/8)\right) \\
=& \frac{1}{4}\left(\frac{1}{2\cdot 1/8} + \frac{1}{2\cdot 3/8} + \frac{1}{2\cdot 5/8} + \frac{1}{2\cdot 7/8}\right) \\
=& \frac{1}{4}\left(\frac{1}{1/4} + \frac{1}{3/4} + \frac{1}{5/4} + \frac{1}{7/4}\right) \\
=& \frac{1}{4}\left(4 + \frac{4}{3} + \frac{4}{5} + \frac{4}{7}\right) \\
=& \frac{176}{105} \\
\approx& 1.68
\end{aligned}$$

Now, since the function is always positive, we can think of this as the area under the curve. However, we still have no idea how good of an approximation this is. In this case, it's horrible since the area is ∞.

Example 4.13. Let's evaluate $\displaystyle\int_0^2 x^2\, dx$ using the midpoint and 6 rectangles.

Recall we did this example using left and right hand endpoints. It took us 100 rectangles to get an estimate of 2.747. We proved the exact area is $\dfrac{8}{3} \approx 2.67$.



1. $\Delta x = \dfrac{2-0}{6} = \dfrac{1}{3}$

2. Find x_i

$$\begin{aligned} x_1 &= 0 + 1 \cdot \frac{1}{3} = \frac{1}{3} \\ x_2 &= 0 + 2 \cdot \frac{1}{3} = \frac{2}{3} \\ x_3 &= 0 + 3 \cdot \frac{1}{3} = \frac{3}{3} \\ x_4 &= 0 + 4 \cdot \frac{1}{3} = \frac{4}{3} \\ x_5 &= 0 + 5 \cdot \frac{1}{3} = \frac{5}{3} \\ x_6 &= 0 + 6 \cdot \frac{1}{3} = \frac{6}{3} \end{aligned}$$

so the midpoints are

$$\begin{aligned} \bar{x_1} &= \frac{1}{6} \\ \bar{x_2} &= \frac{3}{6} \\ \bar{x_3} &= \frac{5}{6} \\ \bar{x_4} &= \frac{7}{6} \\ \bar{x_5} &= \frac{9}{6} \\ \bar{x_6} &= \frac{11}{6} \end{aligned}$$

3. The approximate area is

$$A \approx \frac{1}{3}\left[f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5) + f(x_6)\right]$$
$$A \approx \frac{1}{3}\left[\left(\frac{1}{6}\right)^2 + \left(\frac{3}{6}\right)^2 + \left(\frac{5}{6}\right)^2 + \left(\frac{7}{6}\right)^2 + \left(\frac{9}{6}\right)^2 + \left(\frac{11}{6}\right)^2\right]$$

$$A \approx 2.648$$

So only using 6 rectangles we got much closer to the true area than we did with 100 rectangles! I believe you need somewhere around 190 righthand rectangles for the same approximation as 6 midpoint rectangles!

4.2.2 Properties of the Definite Integral

When we defined the integral
$$\int_a^b f(x)dx,$$

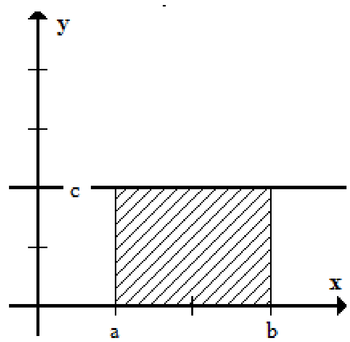
we have so far defined this when $a < b$. However, if $b < a$, the Riemann Sum definition still works, but Δx changes from $\frac{b-a}{n}$ to $\frac{a-b}{n} = \frac{-(b-a)}{n}$. Thus,

$$\int_a^b f(x)dx = -\int_b^a f(x)dx.$$

Further, if $a = b$, then $\Delta x = \frac{b-b}{n} = 0$, and as such
$$\int_b^b f(x)\, dx = 0.$$

Further, to help evaluate a definite integral, we have the following four properties, as long as f and g are continuous functions:

1. $\displaystyle\int_a^b c\,dx = c(b-a)$, for any constant c



2. $\displaystyle\int_a^b (f(x)+g(x))\ dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

3. $\displaystyle\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$ for any constant c

4. $\displaystyle\int_a^b (f(x)-g(x))\ dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

Proving these is a quite simple task. Property 1 follows from the fact that we are integrating a constant height, and our figure would just be a rectangle – area of that is height c times width $b-a$. Properties 2, 3 and 4 are all proven similarly, and we prove property 3 here:

$$\begin{aligned}\int_a^b cf(x)dx &= \lim_{n\to\infty}\sum_{i=1}^n (cf(x_i))\,\Delta x\\ &= \lim_{n\to\infty} c\cdot\sum_{i=1}^n f(x_i)\Delta x\\ &= c\cdot\lim_{n\to\infty}\sum_{i=1}^n f(x_i)\Delta x\\ &= c\cdot\int_a^b f(x)dx\end{aligned}$$

Thus, a constant, but ONLY a constant can be pulled out in front of an integral sign.

Example 4.14. Use the above properties to find

$$\int_2^7 3 - 6x^2 dx.$$

Using the difference property, we have

$$\int_2^7 3 - 6x^2\ dx = \int_2^7 3\ dx - \int_2^7 6x^2\ dx$$

By property 1, we have

$$\int_2^7 3\ dx = 3(7-2) = 15.$$

Also,

$$\int_2^7 6x^2\ dx = 6\int_2^7 x^2\ dx,$$

but to evaluate $\int_2^7 x^2 dx$, we need to treat this as a Riemann Sum, with $\Delta x = \frac{7-2}{n} = \frac{5}{n}$ and $x_i = 2 + \frac{5i}{n}$

$$\begin{aligned}
\int_2^7 x^2 dx &= \lim_{n\to\infty} \sum_{i=1}^n \left(2 + \frac{5i}{n}\right)^2 \cdot \frac{5}{n} \\
&= \lim_{n\to\infty} \frac{5}{n} \sum_{i=1}^n 4 + \frac{20i}{n} + \frac{25i^2}{n^2} \\
&= \lim_{n\to\infty} \frac{5}{n} \left[\sum_{i=1}^n 4 + \sum_{i=1}^n \frac{20i}{n} + \sum_{i=1}^n \frac{25i^2}{n^2}\right] \\
&= \lim_{n\to\infty} \frac{5}{n} \left(4n + \frac{20}{n} \cdot \frac{n(n+1)}{2} + \frac{25}{n^2} \cdot \frac{n(n+1)(2n+1)}{6}\right) \\
&= \lim_{n\to\infty} \frac{5}{n} \left(4n + 10(n+1) + \frac{25(n+1)(2n+1)}{6n}\right) \\
&= 5 \cdot \lim_{n\to\infty} 4 + 10 \cdot \frac{n+1}{n} + 25 \cdot \frac{2n^2+3n+1}{6n^2} \\
&= 5 \cdot \left(4 + 10 + \frac{25}{3}\right) \\
&= \frac{335}{3}
\end{aligned}$$

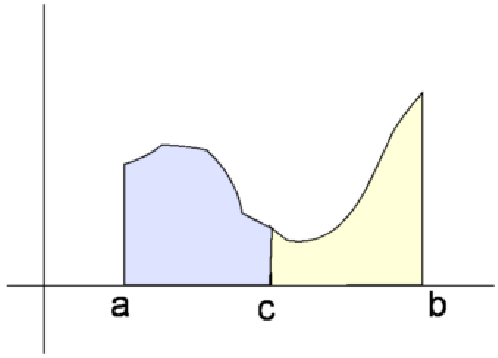
Thus, we have

$$\int_2^7 3 - 6x^2 dx = 15 - 6 \cdot \left(\frac{335}{3}\right) = 15 - 670 = -655.$$

A fifth property involves combining the bounds on two integrals over the same function:

$$\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx,$$

which can easily be seen by the picture below:



After all, if we want to find the area under $f(x)$ from a to b, we can split it somewhere in the middle and add those two areas.

Example 4.15. Suppose that

$$\int_2^{10} f(x)\,dx = 13 \qquad \text{and} \qquad \int_2^7 f(x)\,dx = -3.$$

Find

$$\int_7^{10} f(x) l dx.$$

The answer here lies in a simple application from the following equation:

$$\int_2^{10} f(x)\,dx = \int_2^7 f(x)\,dx + \int_7^{10} f(x)\,dx$$
$$13 = -3 + \int_7^{10} f(x)\,dx$$

Therefore,

$$\int_7^{10} f(x)\,dx = 16$$

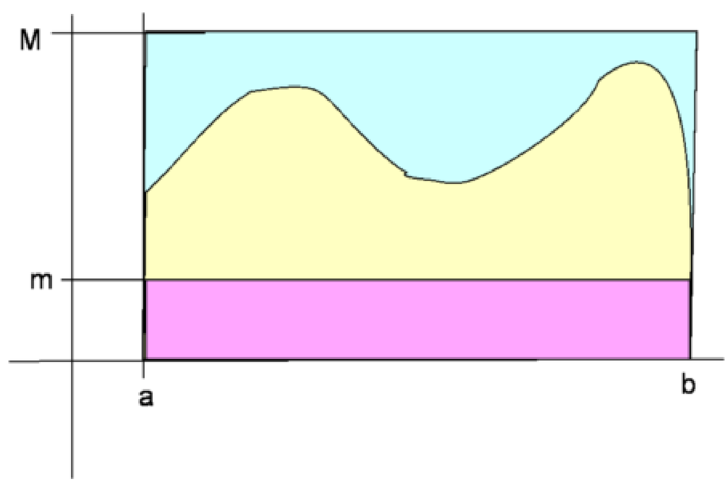
This property is true regardless if $a < c < b$ or not. It doesn't matter at all. Any other configuration would simply be a rearrangement of the variables. However, the following three properties are only true for $a < b$:

6) If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq 0$

7) If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$

8) If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$.

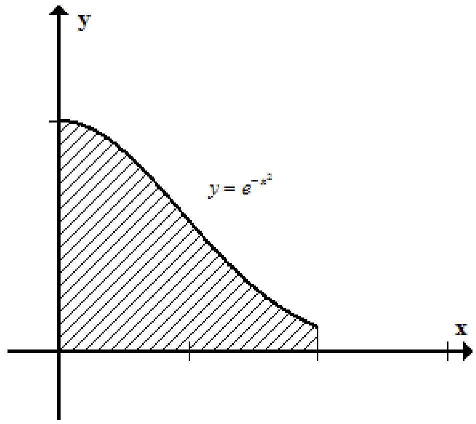
Property 6 implies a positive function gives positive areas. Property 7 says that a bigger (higher) function will have larger area. Duh. Property 8 states that is a function is trapped between two horizontal lines, and as such, the area under the curve is trapped between the areas of two rectangles. None of these really need to be proven, but we we can demonstrate pictorially:



Example 4.16. Estimate the value of

$$\int_0^2 e^{-x^2}\,dx.$$

Take a look at the graph,



From the graph, you can tell that it is a DECREASING function. You can also find this out by the first derivative test. Unfortunately, we do not learn the derivative of exponential functions until next semester.

You can see from the graph, our function has the smallest y value at $x = 2$, which is e^{-4}. It has its highest y-value at $x = 0$, which is $e^0 = 1$. Therefore,

$$e^{-4} < e^{-x^2} < e^0 = 1.$$

Thus, by property 8, we have

$$e^{-4}(2-0) \leq \int_0^2 e^{-x^2} dx \leq 1(2-0)$$

which gives

$$2e^{-4} \leq \int_0^2 e^{-x^2} dx \leq 2.$$