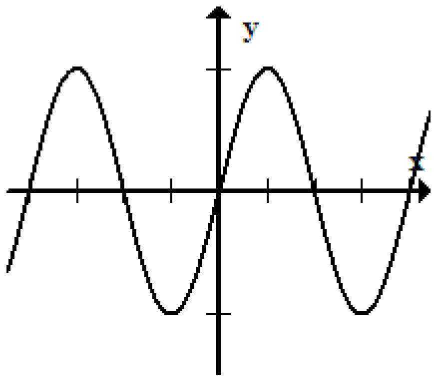
2.4 Derivatives of Trig Functions

Before we go ahead and derive the derivative for $f(x) = \sin(x)$, let's look at its graph and try to graph the derivative first.

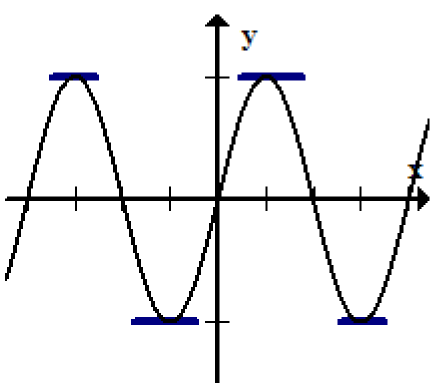


$f(x) = \sin(x)$

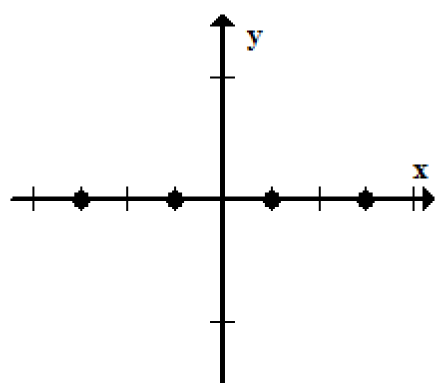
Window $[-2\pi, 2\pi]$, unit - $\pi/2$

1. Remember that the slope on $f(x)$ is the y-value on $f'(x)$.

2. Identify the easy slopes first. I see a slope of 0 at $x = \pi/2$ and $x = 3\pi/2$.

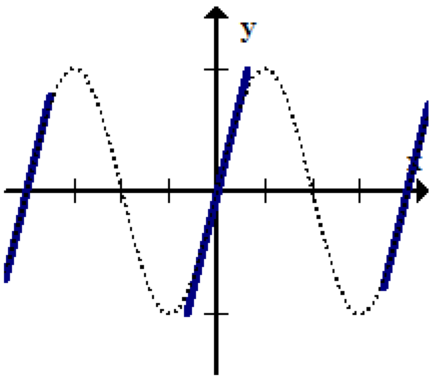


$f(x) = \sin(x)$

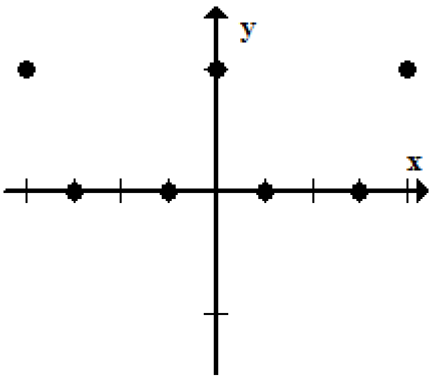


$f'(x) = ?$

3. Estimate some other slopes. For example, at $x = 0$, $x = -2\pi$, and $x = 2\pi$, I'd estimate the slope to be about $m = 1$.

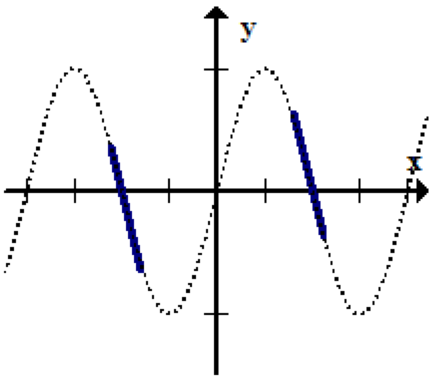


$f(x) = \sin(x)$

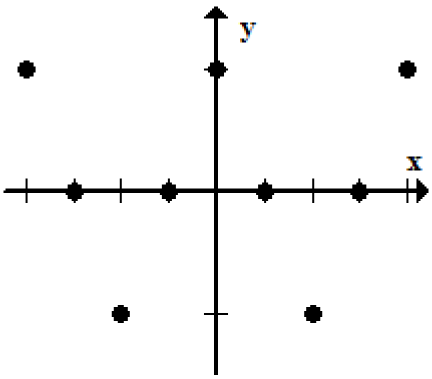


$f'(x) = ?$

4. Estimate some other slopes. For example, at $x = -pi$ and $x = \pi$, I'd estimate the slope to be about $m = -1$.

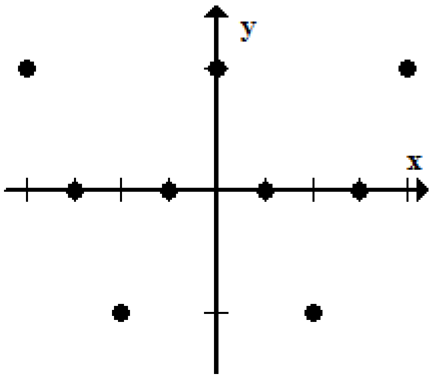


$f(x) = \sin(x)$

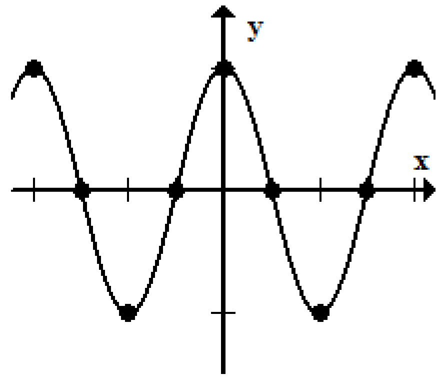


$f'(x) = ?$

5. Ok, that's probably good. Try to connect the dots using a smooth curve.



$f'(x) = ??$



$f'(x) = \cos(x)$

It appears that if $f(x) = \sin(x)$, then $f'(x) = \cos(x)$. Let's see if that's true.

Let $f(x) = \sin(x)$. Using the limit definition of the derivative, we get the

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\sin(x+h)-\sin(x)}{h}
\end{aligned}$$

$$\text{Use} \quad \text{Trig Identity} \ \rightarrow \sin(x+h) = \sin(x)\cos(h) + \sin(h)\cos(x)$$

$$\begin{aligned}
&= \lim_{h\to 0} \frac{\sin(x)\cos(h)+\sin(h)\cos(x)-\sin(x)}{h} \\
&= \text{Rearrange the numerator} \\
&= \lim_{h\to 0} \frac{\sin(x)\cos(h)-\sin(x)+\sin(h)\cos(x)}{h} \\
&= \lim_{h\to 0} \frac{\sin(x)\cos(h)-\sin(x)}{h} + \frac{\sin(h)\cos(x)}{h} \\
&= \lim_{h\to 0} \sin(x)\frac{\cos(h)-1}{h} + \lim_{h\to 0}\cos(x)\frac{\sin(h)}{h} \\
&= \sin(x)\lim_{h\to 0} \frac{\cos(h)-1}{h} + \cos(x)\lim_{h\to 0}\frac{sin(h)}{h} \\
&= \sin(x)\cdot 0 + \cos(x)\cdot 1 \\
&= \cos(x)
\end{aligned}$$

Note: $\lim_{h\to 0} \frac{\sin(h)}{h} = 1$ and $\lim_{h\to 0}\frac{\cos(h)-1}{h}$. We already knew the first limit. The second limit can be found in most calculus texts.

I'm pretty sure we are officially done using the limit definition of the derivative. Yay!!

Let's go through the derivatives of the six trig functions.

2.4.1 Derivatives of Trig Functions

1. $\dfrac{d}{dx}[\sin(x)] = \cos(x)$

2. $\dfrac{d}{dx}[\cos(x)] = -\sin(x)$

3. $\dfrac{d}{dx}[\tan(x)] = \sec^2(x)$

4. $\dfrac{d}{dx}[\sec(x)] = \sec(x)\tan(x)$

5. $\dfrac{d}{dx}[\cot(x)] = -\csc^2(x)$

6. $\dfrac{d}{dx}[\csc(x)] = -\csc(x)\cot(x)$

Example 2.23. Find $\dfrac{d}{dx}[5x^3\cos(x)]$

1. First, identify which rule you should use. I see a power function $5x^3$ and a trig function $\cos(x)$ and they're being multiplied.

2. Use the product rule.

$$\begin{aligned} f'(x) &= 5x^3 \cdot \frac{d}{dx}[\cos(x)] + \cos(x) \cdot \frac{d}{dx}\left[5x^3\right] \\ &= 5x^3 \cdot (-\sin(x)) + \cos(x) \cdot 15x^2 \\ &= -5x^3 \sin(x) + 15x^2 \cos(x) \end{aligned}$$

3. You probably should start simplifying your derivatives. In this case, I'll factor out the GCF.

$$f'(x) = -5x^2(x\sin(x) - 3\cos(x))$$

Example 2.24. If $f(x) = \dfrac{\sqrt{x} - \sec(x)}{\tan(x) + 1400}$

1. First, make sure all non-trig terms are in the proper x^n form.

$$f(x) = \frac{x^{1/2} - \sec(x)}{\tan(x) + 1400}$$

2. Use the Quotient Rule

$$\begin{aligned} f'(x) &= \frac{(\tan(x) + 1400) \cdot \dfrac{d}{dx}\left[x^{1/2} - \sec(x)\right] - (x^{1/2} - \sec(x)) \cdot \dfrac{d}{dx}\left[\tan(x) + 1400\right]}{(\tan(x) + 1400)^2} \\ &= \frac{(\tan(x) + 1400) \cdot \left(\dfrac{1}{2}x^{-1/2} - \sec(x)\tan(x)\right) - (x^{1/2} - \sec(x)) \cdot (\sec^2(x))}{(\tan(x) + 1400)^2} \end{aligned}$$

Example 2.25. Find $\dfrac{d}{dx}\left[4x^3\sin(x)\cot(x)\right]$

1. Notice that $f(x)$ is a product of three functions. We don't have a rule for three functions, only two. This means you need to rewrite this as two functions. I'd do something like this,

$$f(x) = \big(4x^3 \sin(x)\big) \cdot \cot(x)$$

Now it's a product of two functions, $4x^3 \sin(x)$ and $\cot(x)$.

2. Start the product rule. Since this is a complicated problem, show all steps.

$$\begin{aligned}
f'(x) &= \big(4x^3 \sin(x)\big) \cdot \frac{d}{dx}\left[\cot(x)\right] + \cot(x) \cdot \frac{d}{dx}\big[4x^3 \sin(x)\big] \\
&= \big(4x^3 \sin(x)\big) \cdot \big[-\csc^2(x)\big] + \cot(x) \cdot [\text{PRODUCT RULE AGAIN}] \\
&= -\big(4x^3 \sin(x)\big) \cdot \csc^2(x) + \cot(x) \cdot \left[4x^3 \cdot \frac{d}{dx}[\sin(x)] + \sin(x) \cdot \frac{d}{dx}\big[4x^3\big]\right] \\
&= -\big(4x^3 \sin(x)\big) \cdot \csc^2(x) + \cot(x) \cdot \big(4x^3 \cdot \cos(x) + 12x^2 \cdot \sin(x)\big)
\end{aligned}$$

If you want, you can distribute $\cot(x)$ in the second half. It is neither required. The goal of this section is to get you used to the differentiation formulas. The derivatives are messy and complicated. Some really aren't meant to be simplified. Just be careful and take your time. Show as much work as you can. You're less likely to miss something if you're writing everything out. You'll have plenty of practice simplifying derivatives later. Trust me...

2.4.2 Product Rule for Three Functions

$$\frac{d}{dx}\left[f(x) \cdot g(x) \cdot h(x)\right] = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$$