2.6 Implicit Differentiation

The number one thing to remember about implicit functions is that y is a function of x. Sometimes we know what y is and sometimes we don't. For example, if $y = x^2 + x$, we know $y' = 2x + 1$. But what happens when we don't know what y is? Then we just use y' as the derivative. That's it!

1. Explicit functions

   Explicit functions are functions where we know exactly what y is in terms of x.

   (a) $y = x^2 + 5x + 1$ describes y as an explicit function of x

   (b) $y = \sqrt{x^2 - 1}$ describes y as an explicit function of x

   (c) $y = \dfrac{1}{x}$

2. Implicit Functions

   Implicit functions do not tell us what y is in terms of x. But that's ok. Anytime we have to differentiate y when we don't know what it is, just write y'. Here are some examples of implicit functions.

   (a) $x^2 + y^2 = 1$

   (b) $20x - y^2 = 2xy$

(c) $6x - 12y = 3$

(d) $xy = 1$

One thing to note is implicit functions can sometimes be written in an explicit form. For example,

1. $xy = 1$ can be written as $y = \dfrac{1}{x}$

2. $x^2 + y^2 = 1$ can be written as $y = \pm\sqrt{1 - x^2}$

So if you're asked to differentiate an implicit function, you may be able to rewrite the function. However, this is not always the case. For example,

$$20x - y^2 = 2xy$$

cannot be written as an explicit function of x. But that's what this section is all about. How do we differentiate functions defined implicitly?

Before we go ahead and differentiate complicated implicit functions, let's start small.

1. $\dfrac{d}{dx}[y] = y'$

2. $\dfrac{d}{dx}[y^2] = 2y \cdot y'$

   Note, that this is actually a chain rule. Maybe it would help if you look at y^2 as $[y]^2$, where

(a) The outside function is x^2, with $\frac{d}{dx}\left[x^2\right] = 2x$

(b) The inside function is y, with $\frac{d}{dx}[y] = y'$

3. $\frac{d}{dx}\left[y^3\right] = 3y^2 \cdot y'$

4. $\frac{d}{dx}\left[10y^5\right] = 50y^4 \cdot y'$

5. $\frac{d}{dx}\left[\sin(y)\right] = \cos(y) \cdot y'$

6. $\frac{d}{dx}\left[y^{2/3}\right] = \frac{2}{3}y^{-1/3} \cdot y'$

7. $\frac{d}{dx}\left[6xy\right]$

Note, this is actually a product rule of two functions $6x$ and y. Use the product rule as you normally would. When it's time to differentiate y use y'.

$$\frac{d}{dx}\left[6xy\right] = 6x \cdot \frac{d}{dx}[y] + y \cdot \frac{d}{dx}[6x]$$

$$\frac{d}{dx}\left[6xy\right] = 6x \cdot y' + 6y$$

8. $\frac{d}{dx}\left[5x^4y^8\right]$

Again, this is actually a product rule of two functions $5x^4$ and y^8.

$$\begin{aligned}
\frac{d}{dx}\left[5x^4y^8\right] &= 5x^4 \cdot \frac{d}{dx}\left[y^8\right] + y^8 \cdot \frac{d}{dx}\left[5x^4\right] \\
&= 5x^4 \cdot \left[8y^7 \cdot y'\right] + y^8 \cdot \left[20x^3\right] \\
&= 40x^4y^7 + 20x^3y^8
\end{aligned}$$

9. $\dfrac{d}{dx}\left[\tan(x^3y^2)\right]$

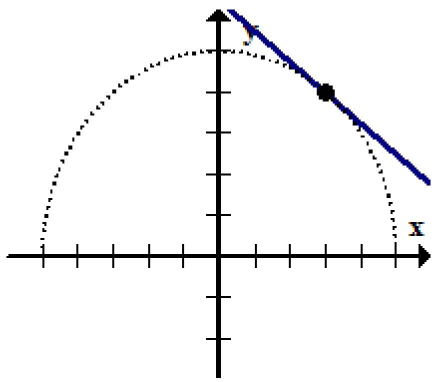
This is a chain rule first, with a product rule inside.

$$\begin{aligned}
\frac{d}{dx}\left[\tan(x^3y^2)\right] &= \sec(x^3y^2) \cdot \frac{d}{dx}\left[x^3y^2\right] \\
&= \sec(x^3y^2) \cdot \left[x^3 \cdot \frac{d}{dx}[y^2] + y^2 \cdot \frac{d}{dx}[x^3]\right] \\
&= \sec(x^3y^2) \cdot \left[x^3 \cdot (2y \cdot y') + y^2 \cdot (3x^2)\right] \\
&= \sec(x^3y^2) \cdot \left[6x^3yy' + 3x^2y^2\right]
\end{aligned}$$

Just to show you that implicit and explicit functions are really doing the same thing, let's differentiate two functions which represent the same half circle. Then find the slope at the point (3,4). Here are the two functions,

$$y = \sqrt{25 - x^2} \text{ and } x^2 + y^2 = 25$$

Here's a picture of what's being asked.



Find the slope of the tangent line shown.

1. $\dfrac{d}{dx}\left[y = \sqrt{25 - x^2}\right]$

   This is a chain rule problem. You should rewrite y as $y = (25 - x^2)^{1/2}$

$$\begin{aligned} y' &= \frac{1}{2}(25 - x^2)^{-1/2} \cdot \frac{d}{dx}\left[25 - x^2\right] \\ &= \frac{1}{2}(25 - x^2)^{-1/2} \cdot [-2x] \\ &= -1x(25 - x^2)^{-1/2} \end{aligned}$$

   Plug in $x = 3$ for the slope at the point $(3, 4)$.

$$y'(3) = -1(3)(25 - (3)^2)^{-1/2} = -3(16)^{-1/2} = \frac{-3}{16^{1/2}} = -\frac{3}{4}$$

2. $\dfrac{d}{dx}\left[x^2 + y^2 = 25\right]$

$$\begin{aligned} 2x + 2y \cdot y' &= 0 \\ \text{Solve} &\quad \text{for} \quad y' \\ 2y \cdot y' &= -2x \\ y' &= -\frac{2x}{2y} \\ y' &= -\frac{x}{y} \end{aligned}$$

Plug in $x = 3$ and $y = 4$.

$$y' = -\frac{3}{4}$$

So you see that if you have the same function, one defined explicitly and the other implicitly, you still get the same answer.

Example 2.36. Find y' when $x^2y^3 = 2x + 1$.

1. Differentiate both sides

$$x^2 \cdot \frac{d}{dx}\left[y^3\right] + y^3 \cdot \frac{d}{dx}\left[x^2\right] = 2$$

$$x^2 \cdot \left[3y^2 \cdot y'\right] + y^3 \cdot [2x] = 2$$

2. Solve for y'

$$\begin{aligned} 3x^2y^2 \cdot y' + 2xy^3 &= 2 \\ 3x^2y^2 \cdot y' &= 2 - 2xy^3 \\ y' &= \frac{2-2xy^3}{3x^2y^2} \end{aligned}$$

Example 2.37. Find $\dfrac{dy}{dx}$ when $y\cos(x) = 1 + \sin(xy)$

1. Differentiate both sides

$$\begin{aligned} y \cdot \frac{d}{dx}[\cos(x)] + \cos(x) \cdot \frac{d}{dx}[y] &= \cos(xy) \cdot \frac{d}{dx}[xy] \\ -y \cdot \sin(x) + \cos(x) \cdot y' &= \cos(xy) \cdot [xy' + 1y] \\ -y\sin(x) + \cos(x) \cdot y' &= x\cos(xy) \cdot y' + y\cos(xy) \end{aligned}$$

2. Solve for y' by bringing all terms with y' to the left side.

$$\cos(x) \cdot y' - x\cos(xy) \cdot y' = y\sin(x) + y\cos(xy)$$

3. Factor out y' and solve

$$\begin{aligned} y'\left[\cos(x) - x\cos(xy)\right] &= y\sin(x) + y\cos(xy) \\ y' &= \frac{y\sin(x) + y\cos(xy)}{\cos(x) - x\cos(xy)} \end{aligned}$$

Example 2.38. Find the slope of the tangent line to the curve $x^2 - 2xy - y^2 = 1$ at (5,2).

1. Differentiate $x^2 - 2xy - y^2 = 1$

$$\begin{aligned} 2x - [2x \cdot y' + 2y] - 2y \cdot y' &= 0 \\ 2x - 2x \cdot y' - 2y - 2y \cdot y' &= 0 \end{aligned}$$

2. Solve for y'

$$\begin{aligned} -2x \cdot y' - 2y \cdot y' &= 2y - 2x \\ y' \cdot (-2x - 2y) &= 2y - 2x \\ y' &= \frac{2y - 2x}{-2x - 2y} \end{aligned}$$

3. To find the slope at (5,2), plug in $x = 5$ and $y = 2$.

$$\begin{aligned} y'(5,2) &= \frac{2(2) - 2(5)}{-2(5) - 2(2)} \\ &= \frac{-6}{-14} \\ &= \frac{3}{7} \end{aligned}$$