1.7 Limit of a Function

We will discuss the following in this section:

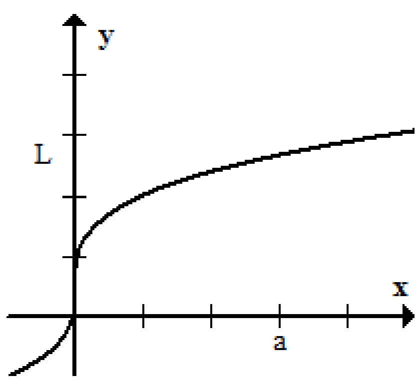
1. Limit Notation

2. Finding a limit numerically

3. Right and Left Hand Limits

4. Infinite Limits

Consider the following graph

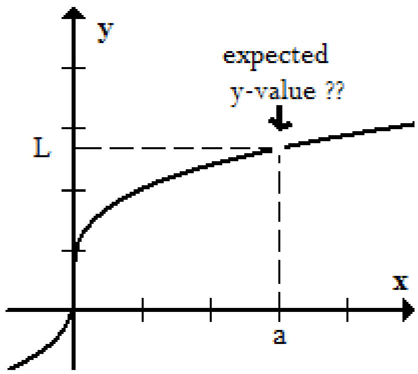


Notation: $\lim_{x \to a} f(x)$, pronounced "limit of $f(x)$ as x approaches a."

- $\lim_{x \to a} f(x) = L$

  means as x gets really close to a (but not equal to a), the corresponding y-values get really close to L.

- Think of the limit as the *expected* y-value.



Example 1.11. Find $\lim_{x\to 2} x^2 - x + 2$

Solution: You'll see that the first step in evaluating limits is simplify plug in what x is approaching. In this case, it's $x = 2$. Don't think too hard about these limits. It's asking, "What would you expect the y-value to be x is really close to 2." I'm hoping your response to this is, "Uh, can't we just plug in $x = 2$." This should make sense. To find the expected y-value, why wouldn't you just plug in $x = 2$?" If you're taking the limit of a nice function, plug in the x-value, and get a nice y-value, then that's the limit.

$$\lim_{x\to 2} x^2 - x + 2 = 2^2 - 2 + 2 = 4$$

But let's pretend like I can't do this. I'd like to stay with this "really close to" concept. What else could I do? Well...if you want to know what the y value would be when x approaches 2, let's plug in x values close to 2. Not too hard, right?

Numerically:

x	y
1	2
1.5	2.75
1.9	3.71
1.99	3.9701
1.999	3.997001
$\downarrow$	$\downarrow$
2	4

Left-Hand Limit

x	y
3	8
2.5	5.75
2.1	4.31
2.01	4.0301
2.001	4.003001
$\downarrow$	$\downarrow$
2	4

Right-Hand Limit

I know I'm skipping ahead a bit, but I think this is as good a place as any to introduce the idea of left and right hand limits. Since a limit is the expected y-value when x approaches a, a left hand limit is the expected y-value when x approaches a from the left (or less than a). A right hand limit is the expected y-value when x approaches a from the right (or greater than a).

So you can see why the left column in the above table is a left hand limit. All the x-values are less than 2. The right column of the table is the right hand limit because all the x-values are greater than 2.

Notation

- Left Hand Limit: $\displaystyle\lim_{x\to a^-} f(x)$ Note the 'exponent' on a

- Right Hand Limit: $\displaystyle\lim_{x\to a^+} f(x)$ Note the 'exponent' on a

The exponent on a is the only difference between the one-sided limits. Be careful! It's easy to miss them or confuse them.

Ok, so why don't we just plug in $x = a$ and get the y-value all the time? It seems like it's the way to go. Let's take a look at the next example.

Example 1.12. Find $\lim_{x\to 0} \dfrac{\sin(x)}{x}$

Notice how we can't just plug in $x = 0$. If we did plug in $x = 0$, we'd get $\dfrac{\sin(0)}{0} = \dfrac{0}{0}$. What is $\dfrac{0}{0}$? Usually $\dfrac{0}{0}$ is telling us something weird is happening with our function. Since we can't just evaluate the function, let's plug some numbers in for x close to 0.

x	y	x	y
.1	.998334	-.1	.998334
.01	.99998	-.01	.99998
.001	.9999999	-.001	.9999999
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
0	1	0	1

As $x \to 0$, we see the y-values are approaching 1. Therefore,

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1$$

Remember this!

Maybe your next question to me is, "Ok, so if I can't plug in $x = a$ (because I'd divide by 0 or something bad like that), I could just plug in x-values close to a. Right?

Wrong. Take a look at our next example.

Example 1.13. $\lim_{x \to 0} \dfrac{\sqrt{x^2+9}-3}{x^2}$

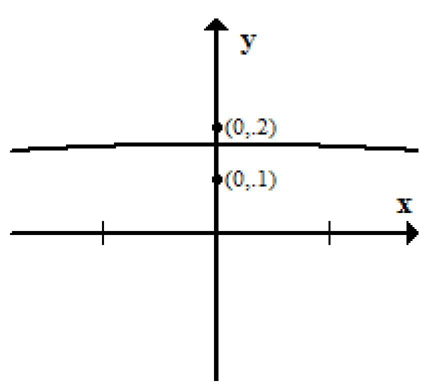
- We can't just plug in $x = 0$ because we'd divide by 0 and that's bad.

- So let's try plugging in x-values close to 0

x	y	x	y
.1	.1666	-.1	.1666
.01	.1667	-.01	.1667
.001	.166667	-.001	.16667
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
	.1666...		.1666...

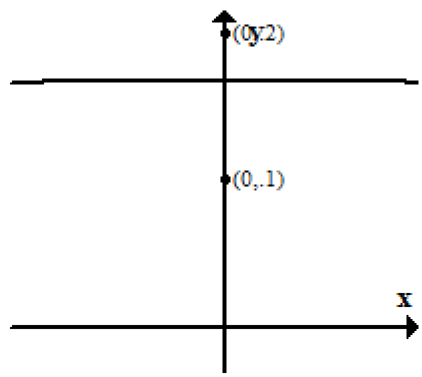
- So based on the table, it seems like $\lim_{x \to 0} \dfrac{\sqrt{x^2+9}-3}{x^2} = .1666...$

- Now let's take a graph of this function.



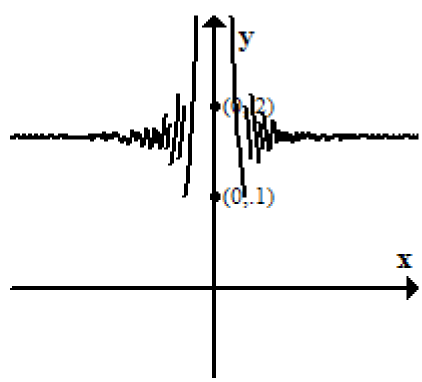
It seems like this graph confirms $\lim_{x \to 0} \dfrac{\sqrt{x^2+9}-3}{x^2} = .1666...$

- Let's dig deeper. Let's zoom in.



We are now at a window of [-.3, .3] x [-.03, .2]. So far so good.

- Zoom in!!



We are at a window of [-.00000001, 0.00000001] x [-.1, .3]. This looks bad. Apparently if we choose x-values extremely close to 0, we start getting some weird y-values. So does that mean the limit isn't .16666...

No. The answer is

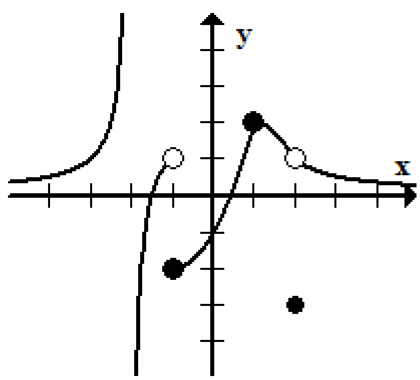
$$\lim_{x\to 0} \frac{\sqrt{x^2+9}-3}{x^2} = .1666...$$

The point I wanted to make is if you plug in numbers close to $x = a$, it's possible you can get something weird (wrong).

Are you sufficiently frustrated yet? I hope so! But that's the fun of it. We're seeing that just simply plugging in x-values close to a may lead to a problem. We need other ways to calculate limits. I'm jumping ahead as that is in the next section. But at this point just know we have methods that take care of these pesky problems.

Before moving on to those wonderful limit methods, let's look at some limits visually.

Example 1.14. Consider the following graph.



Find the following limits:

1. $\lim_{x \to -1^-} f(x)$

   Just follow the graph and approach $x = -1$ from the left. You'd expect the y-value to be 1. The open dot doesn't matter. I still expected it to be $y = 1$.

$$\lim_{x \to -1^-} f(x) = 1$$

2. $\lim_{x \to -1^+} f(x)$

   Follow the graph and approach $x = -1$ from the right. You'd expect the y-value to be -2. The closed dot doesn't matter. I expected the y value to be -2 from the right. Just because it is actually -2 had nothing to do with the limit.

$$\lim_{x \to -1^+} f(x) = -2$$

3. $\lim_{x \to -1} f(x)$

   This is not a one-sided limit. This is a general limit. A general limit exists when it's left and right hand limits exist and they equal equal each other. Since

$$\lim_{x \to -1^-} f(x) = 1 \text{ and } \lim_{x \to -1^+} f(x) = -2, \text{ the general limit}$$

$$\lim_{x \to -1} f(x) \text{ does not exist}$$

4. $\lim_{x \to 1} f(x)$

   Note, this is NOT a one-sided limit. This is a general limit. A general limit only exists if it's left and right hand limits equal each other. So follow the graph to $x = 1$ from both sides. Do they approach the same y-value? Yes they do.

$$\lim_{x \to 1} f(x) = 2$$

5. $\lim_{x \to 2} f(x)$

This is another general limit. Let's take a look at the left and right hand limits. If you follow the graph to $x = 2$, you see an open dot there. Again, it doesn't matter if it's open or closed. The question is, following the graph, where do I expect the y value to be when I get close to $x = 2$. The answer is

$$\lim_{x \to 2} f(x) = 1$$

6. $\lim_{x \to -2^-} f(x)$

Think of $\pm\infty$ as a location or place. We are concerned about the behavior of a function. If there's an asymptote, we don't want to say a limit does not exist. You can see that the function is doing something as it approaches $x = -2$. It's heading up to ∞.

$$\lim_{x \to -2^-} f(x) = \infty$$

7. $\lim_{x \to -2^+} f(x)$

You can see from the graph there is a vertical asymptote at $x = -2$. When following graph to $x = -2$ from the right, you see the graph heads down to $-\infty$.

$$\lim_{x \to -2^+} f(x) = -\infty$$

8. $\lim\limits_{x\to -2} f(x)$

This a general limit. So ask yourself the following question, "Does the left hand limit equal the right hand limit?"

$$\lim_{x\to 2^-} f(x) = \lim_{x\to -2^+} f(x)?$$

From (6) and (7),

$\lim\limits_{x\to -2^-} f(x) = \infty$ and $\lim\limits_{x\to -2^+} f(x) = -\infty$. So the general limit does not exist since the left and right hand limits are not the same.

$$\lim_{x\to -2} f(x) \text{ does not exist}$$

Example 1.15. Consider the following piece-wise function.

$$f(x) = \begin{cases} x^2, & x < -1 \\ -x, & -1 \leq x < 2 \\ \sqrt{x-2} & x \geq 2 \end{cases}$$

Find the following limits.

1. $\lim\limits_{x\to -1^+} f(x) =$

This is a right hand limit. We don't have a graph to look at, but we do have a function. Since it's a right hand limit, it means we're using x-values slightly larger than -1. According to the function, which 'piece' do we use for x-values slightly larger than -1?

$$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} -x = -(-1) = 1$$

2. $\lim\limits_{x\to -1^-} f(x)$

   This is a left hand limit. It means we're using x-values slightly less than -1. According to the function, $f(x) = x^2$ when $x < -1$.

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} x^2 = (-1)^2 = 1$$

3. $\lim\limits_{x\to -1} f(x)$

   This is the general limit. Do the left and right hand limits exist and equal each other? From (1) and (2), we see they do. Therefore,

$$\lim_{x\to -1} f(x) = 1$$

4. $\lim\limits_{x\to 2} f(x)$

   This is a general limit. We haven't computed the left and right hand limits yet. Let's do that now.

   Left-Hand Limit

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^- -x} = -2$$

Right-Hand Limit

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \sqrt{x-2} = 0$$

Since $\lim\limits_{x\to 2^-} f(x) \neq \lim\limits_{x\to 2^+} f(x)$, the general limit

$$\lim_{x\to 2} f(x) \text{ does not exist}$$

Example 1.16. Find $\lim\limits_{x\to 1^-} \dfrac{5+x}{x-1}$ and $\lim\limits_{x\to 1^+} \dfrac{5+x}{x-1}$

Notice that when you plug in $x = 1$, you divide by 0. You'll learn in a little bit that when you plug in an x-value and you get $\dfrac{0}{0}$, it means you have more work to do. These methods we'll learn a bit later. If you plug in an x-value and get a non-zero over 0, like $\dfrac{5}{0}$ or $\dfrac{-3}{0}$, it means the limit is $-\infty$, ∞, or it does not exist.

1. $\lim\limits_{x\to 1^-} \dfrac{5+x}{x-1}$

   Plug in $x = 1$. You get $\dfrac{6}{0}$. So what's the limit? Is it $-\infty$, ∞, or DNE?

   It won't be DNE. When you have a one-sided limit and something like $\dfrac{6}{0}$, it's either $-\infty$ or ∞. We just have to figure out which one. The easiest way is to check the sign of the top and bottom. Choose an x-value close to 1 but less than 1.

$$\lim_{x\to 1^-} \frac{5+x}{x-1} \to \frac{+}{-}$$

The ratio will be $(-)$ negative. Therefore, $\lim_{x\to 1^-} \dfrac{5+x}{x-1} = -\infty$

2. $\lim_{x\to 1^+} \dfrac{5+x}{x-1}$

   We approach this the same way as (1). After plugging in $x = 1$, we get $\dfrac{6}{0}$. So it's either $-\infty$ or ∞. Let's look at the sign of the top and bottom.

$$\lim_{x\to 1^+} \frac{5+x}{x-1} \to \frac{+}{+}$$

   The ratio will be $(+)$ positive. Therefore, $\lim_{x\to 1^+} \dfrac{5+x}{x-1} = \infty$

3. $\lim_{x\to 1} \dfrac{5+x}{x-1}$

   This is the general limit. Since the $\lim_{x\to 1^-} \dfrac{5+x}{x-1} \neq \lim_{x\to 1^+} \dfrac{5+x}{x-1}$

$$\lim_{x\to 1} \frac{5+x}{x-1} \text{ does not exist}$$

Example 1.17. Find $\lim_{x\to -5} \dfrac{x-1}{(5+x)^2}$

You almost always start with just plugging in $x = -5$. If fact, that's exactly what we do here.

$$\frac{-5-1}{5+(-5)^2} = \frac{-6}{0}$$

This means the one-sided limits could be $-\infty$ or ∞. Let's find out.

Left-Hand Limit - Check the sign of the top and bottom.

$$\lim_{x \to -5^-} \frac{x-1}{(5+x)^2} \to \frac{-}{+}$$

Therefore,

$$\lim_{x \to -5^-} \frac{x-1}{(5+x)^2} = -\infty$$

Right-Hand Limit - Check the sign of the top and bottom.

$$\lim_{x \to -5^+} \frac{x-1}{(5+x)^2} \to \frac{-}{+}$$

Therefore,

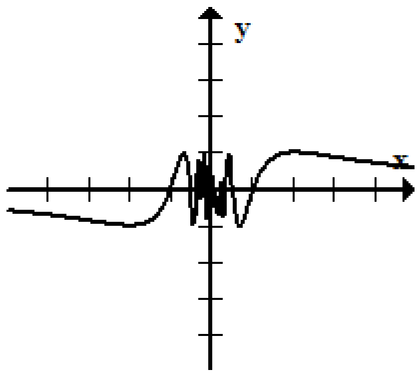
$$\lim_{x \to -5^+} \frac{x-1}{(5+x)^2} = -\infty$$

Since the left and right hand limits equal each other, the answer is

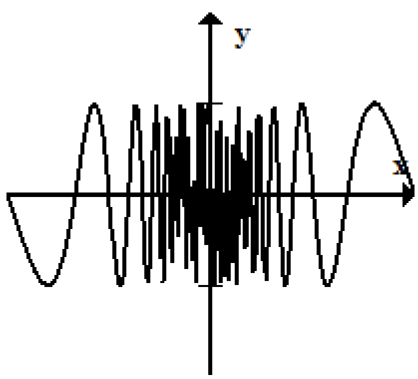
$$\lim_{x \to -5} \frac{x-1}{(5+x)^2} = -\infty$$

Example 1.18. Find $\lim_{x\to 0} \sin\left(\frac{\pi}{x}\right)$

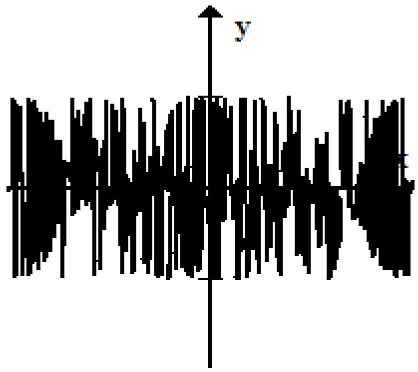
I think looking at the graph could shed a little light.



You can't quite tell what's happening near the origin. Let's zoom in a bit.



Ok, that didn't seem to help. Let's zoom in maybe a bit more. Zoom!



Well that's odd. The window is set now at [-.01, .01] x [-2, 2]. It actually seems like it's getting more chaotic as I zoom it. In fact, that's exactly what it's doing. The limit does not exist because it will never approach a point. Let's try to approach this a different way.

1. Let's look at just $\dfrac{\pi}{x}$.

   As $x \to 0^+$, $\dfrac{\pi}{x} \to \infty$.

2. What happens to sin function when the 'inside' goes to ∞? It oscillates between -1 and 1 forever.

3. That's exactly what is happening to $\lim\limits_{x\to 0} \sin\left(\dfrac{\pi}{x}\right)$. As $x \to 0$, the sin function keeps oscillating between -1 and 1 never approaching a y-value no matter how close we get to $x = 0$.

4. Therefore, $\lim\limits_{x\to 0} \sin\left(\dfrac{\pi}{x}\right)$ does not exist.

Example 1.19. Sketch a graph satisfying all the following conditions.

1. $\lim_{x \to 0} f(x) = 1$

2. $\lim_{x \to 3^-} f(x) = -2$

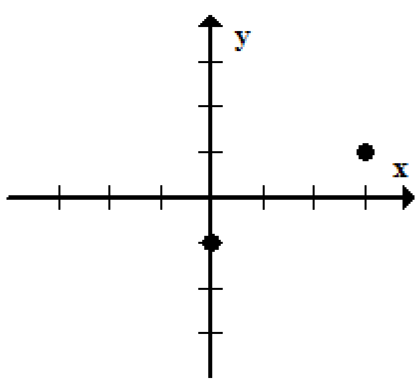
3. $\lim_{x \to 3^+} f(x) = 2$

4. $\lim_{x \to -2} f(x) = -\infty$

5. $f(0) = -1$

6. $f(3) = 1$

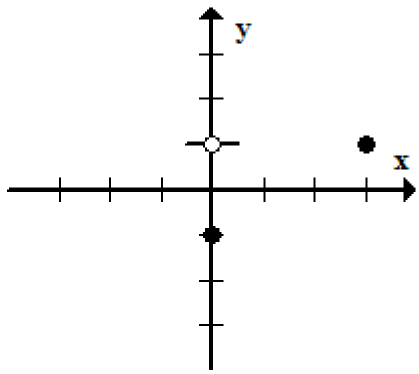
Let's start with (5) and (6) since these are just points on a graph.



The following should be new to you. Since a limit is an expected y-value, we treat the limits as coordinates. BUT! we treat them as open dots since limits don't actually tell us the true y value, just what we would expect.

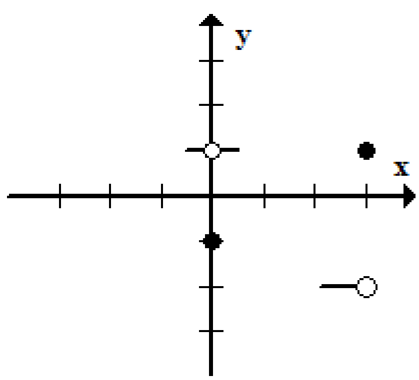
1. $\lim_{x \to 0} f(x) = 1$

   This is a general limit. It means we should expect a y value of 1 when we approach $x = 0$ from the left and right. Notice I put an open dot at (0,1) and drew small lines going left and right from that point.



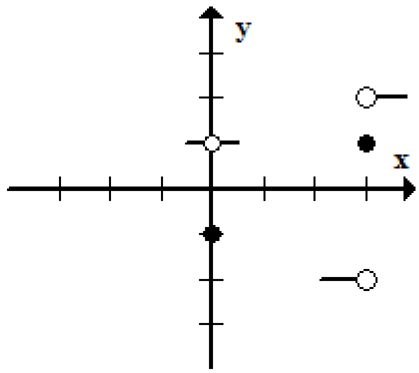
2. $\lim_{x \to 3^-} f(x) = -2$

   This is a left hand limit. We place an open dot at (3,-2) and draw a line approaching this point from the left.



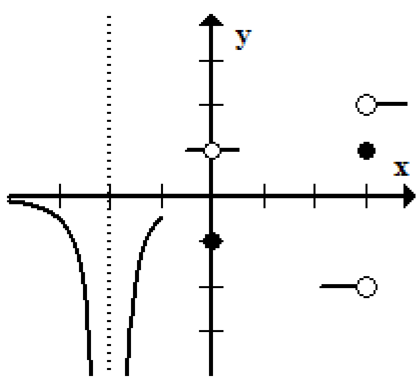
3. $\lim_{x \to 3^+} f(x) = 2$

This is a right hand limit. We place an open dot at (3,2) and draw a line approaching this point from the right.

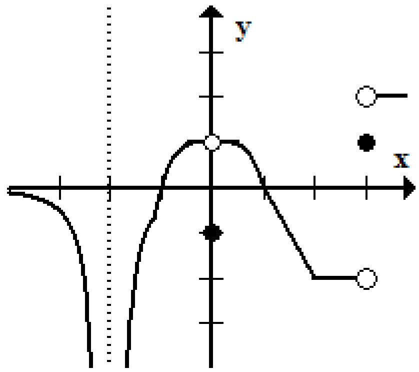


4. $\lim_{x\to -2} f(x) = -\infty$

This is a general limit. Also, this is not a point on the graph but an asymptote at $x = -2$. Since it's a general limit, it means both the left and right hand limits are $-\infty$. It would look something like this.



5. Now, we just connect all the dots to finish the graph.



Wasn't that fun!