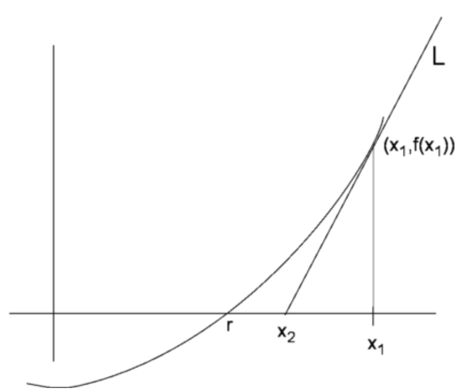
## 3.7 Newton's Method

Finding interest on a loan payment is by no means an easy task. Often, the variable we need to solve for is hiding in the exponent, and if its combined with a sum or difference of multiple terms, no easy way to solve (even taking logs wouldn't help). How do we solve

$$73x(1+2x)^{39} + (1+2x)^{39} - 1,$$

if it doesn't factor (and it doesn't)? Since $f$ is a polynomial (after we substitute $u = 1 + 2x$), it is still a $39^{th}$ degree polynomial and there is no formula to solve greater than a $4^{th}$ degree, and even that is awful. Instead, we can do better, and with less pain, by approximating – for instance, this is what your calculator would do, basically the graph and zoom in method after it plotted a series of points. But how does it do this?

The most commonly used method to find these approximations is called **Newton's Method**, after the cookie guy. No, the apple guy. Yeah, him. Newton's method involves trying to find the root $r$ of a function, and we demonstrate with a picture below:




We make a guess to start, and we call that guess $x_1$. Then, we consider the tangent line $L$ to $y = f(x)$ at this point, $(x_1, f(x_1))$. Now, we need to find the $x$-intercept of this tangent line $L$, the point where $L$ crosses the $x$-axis. We call this point $x_2$ – this is how Newton's method works; the tangent line gives an $x$-intercept closer to the root than our initial guess was. Since the tangent line is close to the curve, the $x$-intercept of the tangent line should be close to the $x$-intercept of the curve, $r$. Even better, since $L$ is just a straight line, we can easily find it's $x$-intercept.

To do this, recall that the slope of the tangent line is the derivative of the function, so the slope of $L$ is $f'(x_1)$. Point-slope form gives the equation

$$y - f(x_1) = f'(x_1)(x - x_1)$$

Since we know that the $x$-intercept of $L$ is $x_2$, we can set the point $(x, y) = (x_2, 0)$:

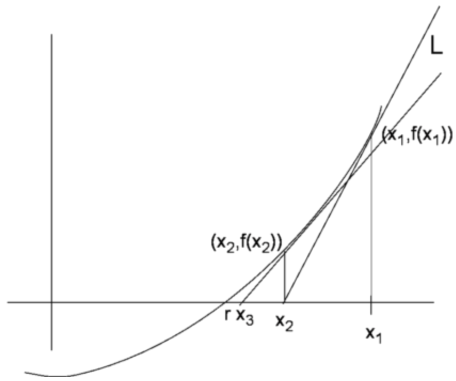
$$0 - f(x_1) = f'(x_1)(x_2 - x_1)$$

For anything but a horizontal tangent line, we have $f'(x_1) \neq 0$, we can solve for $x_2$:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

Then, we can use $x_2$ as an approximation to $r$, repeating this procedure:

Using the tangent line at $(x_2, f(x_2))$ gives a third approximation




$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

And we can keep repeating this process over and over to get a sequence of approximations, $x_1, x_2, x_3, \ldots$. In general, we let the $n^{th}$ approximation be $x_n$ and if $f'(x_n) \neq 0$, then the next approximation is given by

### 3.7.1 Newton's Method Formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

As we let $n \to \infty$, then $x_n \to r$. However, it is possible that the limit does not head to $r$ – such things happen if $f'(x_1) \approx 0$, and $x_2$ is much further away from $r$. In order to fix this issue, we can simply choose a different, and better, approximation for $x_1$.

**Example 3.25.** Let $x_1 = 3$. Find the third approximation to the equation $x^3 - 3x^2 + 6 = 0$.

We apply Newton's Method with

$$f(x) = x^3 - 3x^2 + 6$$

and

$$f'(x) = 3x^2 - 6x$$

We have

$$\begin{aligned}
x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\
x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} \\
&= 3 - \frac{f(3)}{f'(3)} \\
&= 3 - \frac{27 - 27 + 6}{27 - 18} \\
&= 3 - \frac{6}{9} = \frac{7}{3}
\end{aligned}$$

Then, we let $n = 2$ and we get

$$\begin{aligned}
x_3 &= x_2 - \frac{f(x_2)}{f'(x_2)} \\
&= \frac{7}{3} - \frac{f(7/3)}{f'(7/3)} \\
&= \frac{7}{3} - \frac{(7/3)^3 - 3(7/3)^2 + 6}{3(7/3)^2 - 6(7/3)} \\
&= \frac{7}{3} - \frac{64/27}{7/3} = \frac{83}{63}
\end{aligned}$$

While this is the most direct use of Newton's Method, another quite valuable one is to know how far we have to go until we are a certain level of accuracy away from the actual root. If we want to know when we have accuracy to, say, 6 decimal places, we wait until $x_n$ and $x_{n+1}$ agree to 6 decimal places. For instance...

**Example 3.26.** Use Newton's Method to find $\sqrt[3]{2}$ to 6 decimal places.

Finding the decimal value to $\sqrt[3]{2}$ is the same as finding the root to

$$x^3 - 2 = 0.$$

So we let $f(x) = x^3 - 2$ and $f'(x) = 3x^2$, choosing $x_1 = 1$ as our guess, we have

$$\begin{aligned}
x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\
x_{n+1} &= x_n - \frac{x_n^3 - 2}{3x_n^2} \\
x_2 &= 1 - \frac{1^3 - 2}{3(1)^2} = 1 - \frac{-1}{3} = \frac{4}{3} \\
x_3 &= \frac{4}{3} - \frac{(4/3)^3 - 2}{3(4/3)^2} = \frac{91}{72} \\
x_4 &= 1.259933 \\
x_5 &= 1.25992105 \\
x_6 &= 1.2599210499
\end{aligned}$$

Thus, we have $\sqrt[3]{2} \approx 1.259921$.

Now, perhaps needless to say, we did use a calculator for that one. As we probably will for most of our Newton's method problems. Just is the issue with approximating, very difficult to do by hand. Instead, we end this section with two applications.

**Example 3.27.** Use Newton's Method to find the absolute maximum of $f(x) = x\cos(x)$ on $[0, \pi]$.

We first get $f'(x) = \cos(x) - x\sin(x)$. Since $f'(x)$ exists for all $x$, we can find the absolute maximum of $f$ by finding the zeros of $f'$. Thus, here, we need

$$f''(x) = -\sin(x) - \sin(x) - x\cos(x) = -2\sin(x) - x\cos(x).$$

We start by choosing $x = .9$:

$$\begin{aligned}
x_{n+1} &= x_n - \frac{f'(x_n)}{f''(x_n)} \\
x_{n+1} &= x_n - \frac{\cos(x) - x\sin(x)}{-2\sin(x) - x\cos(x)} \\
x_2 &= .9 - \frac{\cos(x) - x\sin(x)}{-2\sin(x) - x\cos(x)} = .8607807 \\
x_3 &= .86033365 \\
x_4 &= .86033359
\end{aligned}$$

Then, since $x_3 \approx x_4 \approx .86033359$, we can evaluate $f(.86033359) = .561096$. Further, since we have a closed interval, we check $f(0) = 0$ and $f(\pi) = \pi(-1) = -\pi$, we have a maximum value at $(.86033359, .561096)$.

**Example 3.28.** Try to find a root of $f(x) = x^3 - 3x + 8 = 0$ starting with $x_1 = 1$.

We have $f'(x) = 3x^2 - 3$, and we have

$$x_{n+1} = x_n - \frac{f'(x_n)}{f''(x_n)} = x_n - \frac{x^3 - 3x + 8}{3x^2 - 3}.$$

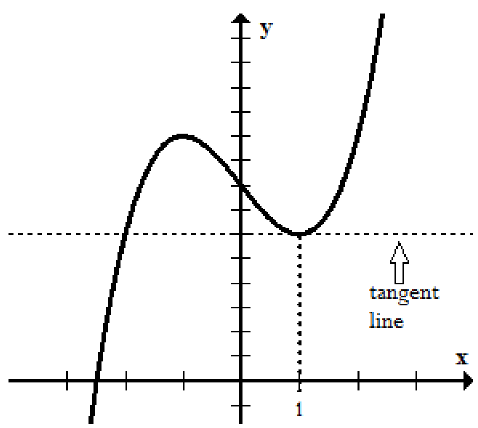
Then

$$x_2 = 1 - \frac{1-3+8}{3-3} = 1 - \frac{6}{0},$$

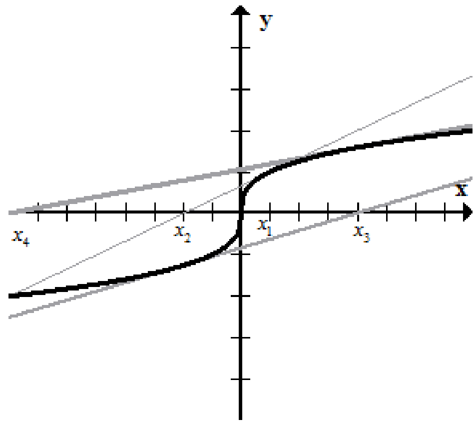
which is quite bad. Yeah, never use a starting approximation which gives a horizontal tangent line. You get a divide by 0 problem, and have to start over.

There are other situations where Newton's Method fails. The previous example failed because you made an initial guess that gave us a horizontal tangent line. That means the tangent line will never go back to the $x$-axis, which means we'll never get our next approximation.

Take a look at the graph. I want to make sure you understand what's happening here.




Another problem is Newton's Method can get stuck in a cycle that does not get you a better approximation. Consider the following equation, $x^{1/3} = 0$. Below is a graph of a few tangent lines. New approximations come from a thicker tangent line. Notice how each tangent line is going away from the root.




The last problem I want to mention is when Newton's Method works but finds the a different root.