3.6   Optimization

Problems in this section are worded very similarly. The objective to these problems is to maximize or minimize a certain quantity. For example,

- How should I create a cylinder from a sheet of metal to maximize the volume?

- How much should I sell this product so I minimize cost?

- Where should an access road connect to the highway to minimize the time it takes from point A to B.

Steps

1. Understand the problem. Can you explain the problem to someone else? If you can't, you don't understand the problem.

    (a) What's known?

    (b) What's unknown?

    (c) What is your objective function?

    (d) Are you minimizing or maximizing?

2. Draw a picture / diagram / chart / etc. Identify what's known and unknown on your picture.

3. Assign a letter to represent the quantity (objective function) to be maximized or minimized. Common letters are

    - $A =$ Area

    - $V =$ Volume

    - $P =$ Perimeter

    - $T =$ Time

    - $P =$ Profit

- Q = Quantity

4. Express your objective function $(A, V, T, ...)$ in terms of the variables, unknowns, knowns, etc.

5. If your objective function is in terms of more than one variable, write all of the variables in terms of one variable.

6. Find the absolute max or min (on the respective domain).

Example 3.20. Find two positive integers whose product is 100 and whose sum is a minimum.

1. Understand the problem? Give yourself some examples so you get an idea of what's going on.

$$10 \cdot 10 = 100 \text{and the sum is } 20$$

$$100 \cdot 1 = 100 \text{and the sum is } 101$$

$$20 \cdot 5 = 100 \text{and the sum is } 25$$

$$25 \cdot 4 = 100 \text{and the sum is } 29$$

Based on this, it appears 10 and 10 do the trick. But these are just some of the examples. I need to use the optimization method from above to prove it's 10 and 10.

2. Let x = one integer and y = the other integer.

Known: $x \cdot y = 100$

Objective: Minimize the sum $S = x + y$

3. Since the objective function is in terms of two variables, I need to write one variable in terms of the other. That's where $x \cdot y = 100$ comes in.

$$y = \frac{100}{x}$$

The new objective function is

$$\text{Minimize } S = x + \frac{100}{x}$$

4. To find the minimum of $S = x + \frac{100}{x}$, we need to find S'

$$S' = 1 - \frac{100}{x^2}$$

Now, we find the critical values. One critical value is $x = 0$, but we don't count it because we know our integers have to be positive.
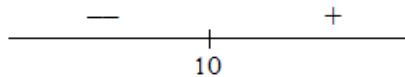
$$1 - \frac{100}{x^2} = 0$$

$$1 = \frac{100}{x^2}$$

$$x^2 = 100$$

$$x = \pm 10$$

We only take $x = 10$ as a critical value because $x = -10$ is not positive.

5. Just to make sure we have our minimum at $x = 10$, let's use the number line to check.

So we have a local minimum at $x = 10$. We confirmed our initial guess that the numbers are 10 and 10. But actually, at this point, we've only confirmed one integer is 10. To find the other integer y, use

$$y = \frac{100}{x}$$

$$y = \frac{100}{10} = 10$$

Therefore, two numbers that multiply 100 and whose sum is at a minimum is 10 and 10.

Example 3.21. Suppose I want to enclose off an area for a garden. It will be up against my house, so I don't have to fence the side that's against the house. I have 24 feet of fencing. How should I construct this rectangular garden so I get the largest area?

1. Let's start with a picture. This should help you understand the problem.
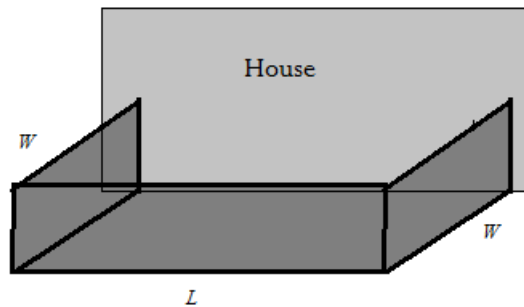
   Known: $2W + L = 24$

   Objective: Maximize $A = L \cdot W$

2. Notice that there are two variables L and W. We will use $2W + L = 24$ to write L in terms of W.

$$2W + L = 24$$

$$L = 24 - 2W$$

Our new objective is

$$\text{Maximize } A = (24 - 2W) \cdot W = 24W - 2W^2$$

3. To maximize $A = 24W - 2W^2$, we need to find A'

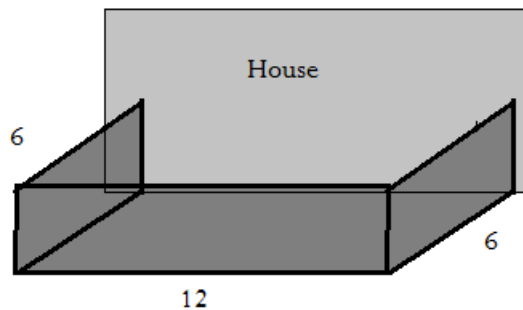$$A' = 24 - 4W$$

To find the critical values, set $A' = 0$

$$24 - 4W = 0$$

gives us a critical value at $W = 6$. Verify by using the number line and we'll find that $W = 6$ is a local maximum.

If $W = 6$, we use $L = 24 - 2W$ to find L.

$$L = 24 - 2(6) = 12$$

To get the largest area, I should fence off a rectangular region with the dimensions $W = 6$ and $L = 12$

Example 3.22. Your task is to build an access road joining a small village to a highway that enables drivers to reach a large city in the shortest time. A picture is given below. How should this be done is the speed limit on the access road is 35 mph and the speed limit on

the highway is 55 mph. The shortest possible distance from the village to the highway is 15 miles and the city is 45 miles down the highway.
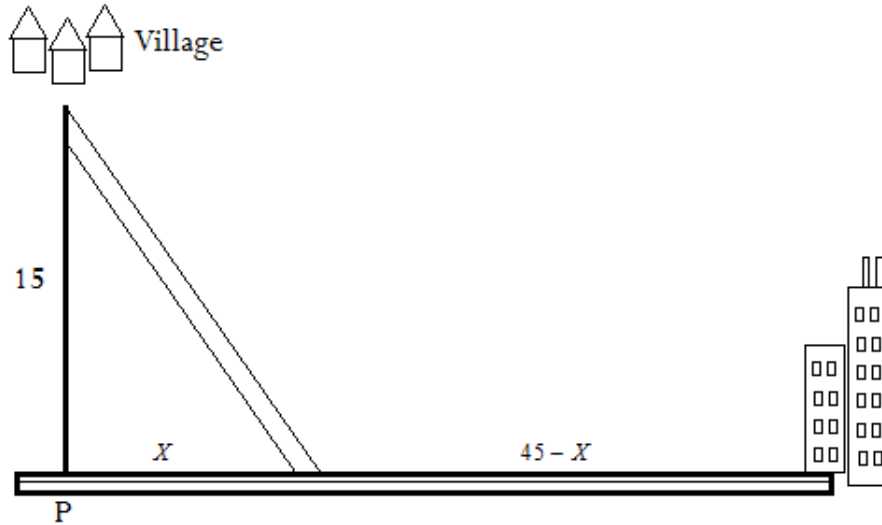
1. Let $x =$ the distance down the highway where the access road connects to the highway. As a function of x, the length of the access road is

$$\sqrt{15^2 + x^2}$$

   The distance remaining to the city is $45 - x$.

   The total distance traveled is

$$\sqrt{15^2 + x^2} + (45 - x)$$

2. But we are concerned with minimizing the time it takes to get from the village to the city. So we need a formula for time traveled. The formula for time is

$$t = \frac{\text{distance}}{\text{speed}} = \frac{d}{s}$$

Since the speed is different for the two roads, we need to split it up into two times, one for the access road, and one for the highway.

3. Time traveled on the access road is $\dfrac{\sqrt{15^2 + x^2}}{35}$.

4. Time traveled on the highway is $\dfrac{45 - x}{55}$

5. Our objective is to

$$\text{Minimize } T(x) = \frac{\sqrt{15^2 + x^2}}{35} + \frac{45 - x}{55}$$

over the interval $0 \leq x \leq 50$. Keep in mind, it's very possible that to minimize time the access road can connect to the highway along the shortest possible distance ($x = 0$) or the access road can go all the way to the city ($x = 50$).

6. Let's go ahead and find $T'(x)$. Let's rewrite $T(x)$ so it's easier to differentiate,

$$T(x) = \frac{1}{35}\left(15^2 + x^2\right)^{1/2} + \frac{1}{55}(45 - x)$$

$$
\begin{aligned}
T'(x) &= \frac{1}{35} \cdot \frac{1}{2}(15^2 + x^2)^{-1/2} \cdot (2x) + \frac{1}{55} \cdot (-1) \\
&= \frac{x}{30(15^2 + x^2)^{1/2}} - \frac{1}{55}
\end{aligned}
$$

7. To find the critical values, set $T'(x) = 0$.

$$\frac{x}{30(15^2 + x^2)^{1/2}} - \frac{1}{55} = 0$$

$$\frac{x}{30(15^2 + x^2)^{1/2}} = \frac{1}{55}$$

cross multiply

$$55x = 30(15^2 + x^2)^{1/2}$$

$$11x = 6(15^2 + x^2)^{1/2}$$

square both sides

$$121x^2 = 36(15^2 + x^2)$$

$$121x^2 = 8100 + 36x^2$$

$$85x^2 = 8100$$

$$x^2 = \frac{8100}{85}$$

$$x = 9.76$$

8. Since we're looking for the ABSOLUTE minimum, we check the time traveled at the endpoints, $x = 0$ and $x = 50$, and the critical value at $x = 9.76$.
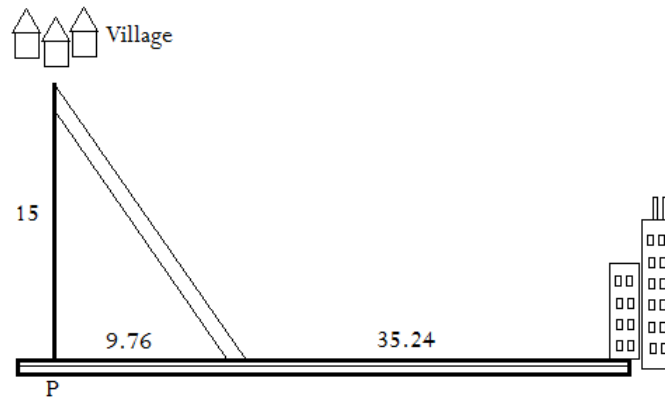
$$T(0) = \frac{1}{35}\left(15^2 + 0^2\right)^{1/2} + \frac{1}{55}(45 - 0) = 1.25 \text{ hours}$$

$$T(9.76) = \frac{1}{35}\left(15^2 + 9.76^2\right)^{1/2} + \frac{1}{55}(45 - 9.76) = 1.15 \text{ hours}$$

$$T(45) = \frac{1}{35}\left(15^2 + 45^2\right)^{1/2} + \frac{1}{55}(45 - 45) = 1.36 \text{ hours}$$

The time traveled is minimized if the access road is connected to the highway 9.76 miles from P.

Village

15

9.76   35.24

P

Example 3.23. Suppose you have a cylinder with a volume of 355 cm^3 or 21.66 in^3. What should be the radius and height of the cylinder so that it uses the least amount of material.

1. Before we begin, using the least amount of material means minimizing surface area of the material used.

2. Let's take a look at some examples.

6.89 in

689.46 in

0.77 in

r = 0.1
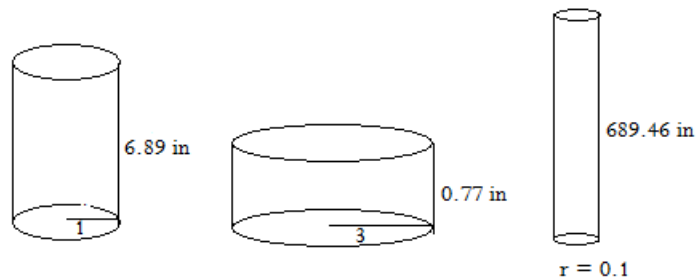
3. So we need two formulas. One is the formula for the volume of a cylinder. The second is the formula for the surface area of a cylinder.

$$\text{Volume: } V = \pi r^2 h$$

$$\text{Surface Area: } S = 2\pi r h + 2\pi r^2$$

4. Our objective is to minimize $S = 2\pi rh + 2\pi r^2$

5. The objective function is in terms of two variables, r and h. We need to rewrite one in terms of the other. Since r shows up more often in the formula, let's rewrite h in terms of r. We will use the volume formula to do it.

$$V = 21.66 = \pi r^2 h$$

$$h = 21.66/(\pi r^2)$$

6. So our new objective is to minimize

$$S = 2\pi r \cdot \left(\frac{21.66}{\pi r^2} \right) + 2\pi r^2 = \frac{43.32}{r} + 2\pi r^2$$

7. Let's find S' and set it equal to 0.

$$S' = -\frac{43.32}{r^2} + 4\pi r$$

Let's set $S' = 0$

$$
\begin{aligned}
-\frac{43.32}{r^2} + 4\pi r &= 0 \\
4\pi r &= \frac{43.32}{r^2} \\
4\pi r^3 &= 43.32 \\
r^3 &= 3.45 \\
r &= 1.51
\end{aligned}
$$

8. We minimize the material used if the cylinder has a radius of 1.511 inches. You can verify it's a local minimum by using the number line. Now we find h

$$h = \frac{21.66}{\pi(1.51)^2}$$

$$h = 3.02 \text{ inches}$$

By the way, 21.66 in^3 is the volume of a soda can. The dimensions of a soda can differ slightly from the dimensions we found that minimizes the material used. But there are a couple of things to note. Soda cans are not perfectly cylindrical. We found the radius should be 1.51 in, but a soda can is actually about 1.3 in. It's possible a soda can has a smaller radius so it's easier to hold.

Example 3.24. During the summer Samantha makes and sells necklaces. Last summer she sold each one for \$10 and averaged 20 sales per day. When she increased the price by \$1, the sales decreased by two per day.

1. Find the demand function.

2. If material costs \$6, what should the selling price by to maximize profit?

These are good problems because in real life trying to maximize profit consists of some trial and error. After a couple of different scenarios, you try to find which one makes you the most money.

Note:

1. Profit = Revenue - Cost

2. Revenue $= x \cdot p(x)$, where $x =$ the number sold and $p(x)$ is the price.

3. The demand function is $p(x)$. The price of a necklace depends on the number of necklaces sold per day.

So how do we find the demand function, $p(x)$. Notice that we have two pieces of data. We know $p(x)$ must go through the points (20, 10) and (18, 11). So we can estimate $p(x)$ by finding the equation of a line that passes through these two points.

$$\text{Slope: } m = \frac{11 - 10}{18 - 20} = -\frac{1}{2}$$

Using the point-slope formula

$$
\begin{aligned}
y - y_1 &= m(x - x_1) \\
y - 10 &= -\frac{1}{2}(x - 20) \\
y &= -\frac{1}{2}x + 20
\end{aligned}
$$

Therefore, the demand function is $p(x) = -\frac{1}{2} + 20$. What does this mean? It means for every extra 2 necklaces Samantha sells, the price must go down by $1.

Let's move to maximizing profit.

$$
\begin{aligned}
P(x) &= R(x) - C(x) \\
P(x) &= x \cdot p(x) - C(x) \\
P(x) &= x \cdot \left(-\frac{1}{2}x + 20\right) - (6x) \\
P(x) &= -\frac{1}{2}x^2 + 20x - 6x \\
P(x) &= -\frac{1}{2}x^2 + 14x
\end{aligned}
$$

Note, $C(x) = 6x$ because each necklace costs $6 to make.

To maximize $P(x)$, we need to find $P'(x)$.

$$P'(x) = -x + 14$$

Setting $P'(x) = 0$, we find $x = 14$. You can verify this is the maximum by using the number line.

So to maximize profit, Samantha needs to sell 14 necklaces. The price per necklace is

$$P(14) = -\frac{1}{2}(14) + 20 = 13$$

Conclusion: Samantha needs to sell 14 necklaces at \$13 to maximize profit.